

Remark on stabilization of second order evolution equations by unbounded dynamic feedbacks and applications

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Abstract

In this paper we consider second order evolution equations with unbounded dynamic feedbacks. Under a regularity assumption we show that observability properties for the undamped problem imply decay estimates for the damped problem. We consider both uniform and non uniform decay properties.

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1 Introduction

Let X be a complex Hilbert space with norm and inner product denoted respectively by $\|\cdot\|_X$ and $\langle \cdot, \cdot \rangle_{X,X}$. Let A be a linear unbounded positive self-adjoint operator which is the Friedrichs extension of the triple (X, V, a) , where a is a closed quadratic form with domain V dense in X . Note that by definition $\mathcal{D}(A)$ (the domain of A) is dense in X and $\mathcal{D}(A)$ equipped with the graph norm is a Hilbert space and the embedding $\mathcal{D}(A) \subset X$ is continuous. Further, let U be a complex Hilbert space (which will be identified with its dual space) with norm and inner product respectively denoted by $\|\cdot\|_U$ and $\langle \cdot, \cdot \rangle_{U,U}$ and let $B \in \mathcal{L}(U, V')$, where V' is the dual space of V obtained by means of the inner product in X . Consider the system

$$(1) \quad \begin{cases} x''(t) + Ax(t) + Bu(t) = 0, & t \in [0, +\infty) \\ \rho u'(t) - \widehat{C}u(t) - B^*x'(t) = 0, & t \in [0, +\infty) \\ x(0) = x_0, x'(0) = y_0, u(0) = u_0, \end{cases}$$

with ρ a scalar parameter. By replacing ρ by 0 and $-\widehat{C}$ by the identity in system (1) we obtain the system whose stability was studied in [4].

In this paper we are interested in studying the stability of linear control problems coming from elasticity which can be written as

$$(2) \quad \begin{cases} x''(t) + Ax(t) + Bu(t) = 0, & t \in [0, +\infty) \\ u'(t) - \widehat{C}u(t) - B^*x'(t) = 0, & t \in [0, +\infty) \\ x(0) = x_0, x'(0) = y_0, u(0) = u_0, \end{cases}$$

where $x : [0, +\infty) \rightarrow X$ is the state of the system, $u \in L^2(0, T; U)$ is the input function and \widehat{C} is a m -dissipative operator on U . We denote the differentiation with respect to time by $'$.

The aim of this paper is to give sufficient conditions leading to the uniform or non uniform stability of the solutions of the corresponding closed loop system.

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The second equation of the considered system describes a dynamical control in some models. Some systems that can be covered by the formulation (2) are for example the hybrid systems.

Let us finish this introduction with some notation used in the remainder of the paper: the notation $A \lesssim B$ and $A \sim B$ means the existence of positive constants C_1 and C_2 , which are independent of A and B such that $A \leq C_2 B$ and $C_1 B \leq A \leq C_2 B$.

2 Well-posedness results

In order to study the system (2) we use a reduction order argument. First, we introduce the Hilbert space $\mathcal{H} = V \times X \times U$ equipped with the scalar product

$$\langle z, \tilde{z} \rangle_{\mathcal{H}, \mathcal{H}} = a(x, \tilde{x}) + \langle y, \tilde{y} \rangle_{X, X} + \langle u, \tilde{u} \rangle_{U, U}, \quad \forall z, \tilde{z} \in \mathcal{H}, z = (x, y, u), \tilde{z} = (\tilde{x}, \tilde{y}, \tilde{u}).$$

Then we consider the unbounded operator

$$\begin{aligned} \mathcal{A}_d : \mathcal{D}(\mathcal{A}_d) &\longrightarrow \mathcal{H} \\ z = (x, y, u) &\longmapsto \mathcal{A}_d z = (y, -Ax - Bu, B^*y + \hat{C}u), \end{aligned}$$

where

$$\mathcal{D}(\mathcal{A}_d) = \{(x, y, u) \in V \times V \times D(\hat{C}), Ax + Bu \in X\}.$$

So the system (2) is formally equivalent to

$$(3) \quad z'(t) = \mathcal{A}_d z(t), z(0) = z_0,$$

where $z_0 = (x_0, y_0, u_0)$.

Proposition 2.1. *The operator \mathcal{A}_d is an m -dissipative operator on \mathcal{H} and thus it generates a C_0 -semigroup.*

Proof.

$$\begin{aligned} \langle \mathcal{A}_d z, z \rangle_{\mathcal{H}, \mathcal{H}} &= a(y, x) - \langle Ax + Bu, y \rangle_{X, X} + \langle B^*y + \hat{C}u, u \rangle_{U, U} \\ &= a(y, x) - a(x, y) - \langle Bu, y \rangle_{V', V} + \langle B^*y, u \rangle_{U, U} + \langle \hat{C}u, u \rangle_{U, U} \\ &= a(y, x) - a(x, y) + \langle \hat{C}u, u \rangle_{U, U}. \end{aligned}$$

Taking the real part of the above identity we get (5) since \hat{C} is dissipative. Hence \mathcal{A}_d is dissipative.

We would like to show that there exists $\lambda > 0$ such that $\lambda I - \mathcal{A}_d$ is surjective. Let $\lambda > 0$ be given. Clearly, we have $\lambda \notin \sigma(\hat{C})$. For $(f, g, h) \in \mathcal{H}$, we look for $(x, y, u) \in \mathcal{D}(\mathcal{A}_d)$ such that

$$(\lambda I - \mathcal{A}_d) \begin{pmatrix} x \\ y \\ u \end{pmatrix} = \begin{pmatrix} f \\ g \\ h \end{pmatrix},$$

i.e. we are searching for $x \in V, y \in V, u \in D(\hat{C})$ satisfying

$$\begin{aligned} \lambda x - y &= f \\ \lambda^2 x + Ax + Bu &= g + \lambda f \\ (\lambda I - \hat{C})u - B^*y &= h. \end{aligned}$$

By Lax-Milgram lemma there exists a unique $x \in V$ such that

$$\left(\lambda^2 + A + \lambda B(\lambda I - \hat{C})^{-1} B^* \right) x = g + \lambda f + B(\lambda I - \hat{C})^{-1} (B^* f - h).$$

In fact, we have $\lambda^2 + A + \lambda B(\lambda I - \hat{C})^{-1} B^* \in \mathcal{L}(V, V')$, $g + \lambda f + B(\lambda I - \hat{C})^{-1} (B^* f - h) \in V'$ and

$$\Re \left\langle \left(\lambda^2 + A + \lambda B(\lambda I - \hat{C})^{-1} B^* \right) x, x \right\rangle_{V', V} \geq \langle Ax, x \rangle_{V', V},$$

since

$$\begin{aligned}\Re \left\langle B(\lambda I - \widehat{C})^{-1} B^* x, x \right\rangle_{V', V} &= \Re \left\langle u, (\lambda I - \widehat{C}) u \right\rangle_{U, U} \\ &= \lambda \|u\|^2 - \Re \left\langle u, \widehat{C} u \right\rangle_{U, U} \geq 0,\end{aligned}$$

with $u = (\lambda I - \widehat{C})^{-1} B^* x$, i.e. the coercivity property is satisfied.

Define

$$u = (\lambda I - \widehat{C})^{-1} (h + B^* (\lambda x - f)),$$

by choosing $y = \lambda x - f$ we deduce the surjectivity of $\lambda I - A$. Finally, we conclude that $\lambda I - A$ is bijective, for all $\lambda > 0$. □

Now, we are able to state the following existence result of problem (3).

Proposition 2.2. (i) For an initial datum $z_0 \in \mathcal{H}$, there exists a unique solution $z \in C([0, +\infty), \mathcal{H})$ to system (3). Moreover, if $z_0 \in \mathcal{D}(\mathcal{A}_d)$, then

$$(4) \quad z \in C([0, +\infty), \mathcal{D}(\mathcal{A}_d)) \cap C^1([0, +\infty), \mathcal{H}).$$

(ii) For each $z_0 \in \mathcal{D}(\mathcal{A}_d)$, the energy $E(t)$ of the solution z of (3), defined by

$$E(t) = \frac{1}{2} \|z(t)\|_{\mathcal{H}}^2,$$

satisfies

$$(5) \quad E'(t) = \Re \langle \widehat{C} u(t), u(t) \rangle \leq 0,$$

therefore the energy is non-increasing.

Moreover, we have the following estimate

$$(6) \quad E(0) - E(t) = - \int_0^t \Re \langle \widehat{C} u(s), u(s) \rangle ds \leq \frac{1}{2} \|z_0\|_{\mathcal{H}}^2, \quad \forall t \in [0, +\infty), \forall z_0 \in \mathcal{H}.$$

Proof. (i) is a direct consequence of Lumer-Phillips theorem (see [7]).

(ii) For an initial datum in $\mathcal{D}(\mathcal{A}_d)$ from (4), we know that u is of class C^1 in time, thus we can derive the energy $E(t)$, and using Proposition 2.1 we obtain:

$$E'(t) = \Re \langle z', z \rangle_{\mathcal{H}, \mathcal{H}} = \Re \langle \mathcal{A}_d z, z \rangle_{\mathcal{H}, \mathcal{H}} = \Re \langle \widehat{C} u, u \rangle.$$

Hence the energy is non-increasing. Finally (6) is a direct consequence of (5). □

Assume that \widehat{C} can be written as $\widehat{C} = -C - DD^*$ where C is a skew-adjoint operator on U , $D \in \mathcal{L}(W, (D(C))')$, and W is supposed to be a Hilbert subspace of U identified with its dual, thus $D^* \in \mathcal{L}(D(C), W)$. Denote by \mathcal{A}_c the operator obtained by replacing \widehat{C} by $-C$ in the expression \mathcal{A}_d . We can easily check that \mathcal{A}_c is closed anti-symmetric, m-dissipative operator whose opposite $-\mathcal{A}_c$ is also maximal dissipative, therefore \mathcal{A}_c is skew-adjoint and generates a unitary group. Denote by \mathcal{A}_r the operator

$$\mathcal{A}_r : (x, y, u) \in \mathcal{H} \mapsto (0, 0, -DD^* u),$$

it is easy to see that \mathcal{A}_r is dissipative and $\mathcal{A}_d = \mathcal{A}_c + \mathcal{A}_r$. Note that the energy satisfies:

$$(7) \quad E'(t) = -\|D^* u(t)\|_W^2.$$

3 Some regularity results

Let $T > 0$ be fixed and $u \in L^2(0, T; U)$. Consider the evolution problem

$$(8) \quad z_2'(t) = \mathcal{A}_c z_2(t) + \mathcal{A}_r z(t), z_2(0) = 0, t \in [0, T],$$

where $\mathcal{A}_r z(t) = -(0, 0, DD^*u(t))$.

Lemma 3.1. *Suppose that $D \in \mathcal{L}(U)$. Then problem (8) admits a unique solution $z_2(t) = (x_2(t), y_2(t), u_2(t))$ such that*

$$u_2 \in L^2(0, T; U),$$

satisfying the following estimate

$$(9) \quad \|D^*u_2\|_{L^2(0, T; U)} \leq c\|D^*u\|_{L^2(0, T; U)},$$

where c is a positive constant.

Proof. Clearly $\mathcal{A}_r z(t) = -(0, 0, DD^*u(t)) \in C^1(0, T; \mathcal{H})$, and since \mathcal{A}_c generates a unitary group $e^{\mathcal{A}_c \cdot}$ on \mathcal{H} , then (8) admits a unique solution given by

$$z_2(t) = \int_0^t e^{\mathcal{A}_c(t-s)} \mathcal{A}_r z(s) ds = \int_0^t e^{\mathcal{A}_c(s)} \mathcal{A}_r z(t-s) ds, \forall t \in [0, T].$$

Moreover $D \in \mathcal{L}(U)$ and

$$\begin{aligned} \|u_2\|_{L^2(0, T; U)}^2 &= \int_0^T \|u_2(t)\|_U^2 dt \\ &\leq \int_0^T \|z_2(t)\|_{\mathcal{H}}^2 dt \\ &\leq \int_0^T \left\| \int_0^t e^{\mathcal{A}_c s} \mathcal{A}_r z(t-s) ds \right\|_{\mathcal{H}}^2 dt \\ &\leq \int_0^T \left(\int_0^t \|e^{\mathcal{A}_c s}\| \|\mathcal{A}_r z(t-s)\|_{\mathcal{H}} ds \right)^2 dt \\ &\leq \int_0^T \left(\int_0^t \|\mathcal{A}_r z(s)\|_{\mathcal{H}} ds \right)^2 dt \\ &\leq \int_0^T \left(\int_0^T \|\mathcal{A}_r z(s)\|_{\mathcal{H}} ds \right)^2 dt \\ &\leq \int_0^T \left(\int_0^T \|DD^*u(s)\|_U ds \right)^2 dt \\ &\leq \int_0^T (\int_0^T 1^2 ds) (\int_0^T \|DD^*u(s)\|_U^2 ds) dt \\ &\leq \int_0^T T \|D\|^2 \|D^*u\|_{L^2(0, T; U)}^2 dt \\ &\leq T^2 \|D\|^2 \|D^*u\|_{L^2(0, T; U)}^2. \end{aligned}$$

Consequently, as $\|D^*u_2\|_{L^2(0, T; U)} \leq \|D^*\| \|u_2\|_{L^2(0, T; U)}$, (9) holds with the constant $T\|D\|\|D^*\|$. \square

4 Uniform stability

In this section, we give sufficient and necessary condition which lead to uniform stability of system (3). We first introduce the conservative system associated with the initial problem (2) as

$$(10) \quad \begin{cases} x''(t) + Ax(t) + Bu(t) = 0, & t \in (0, +\infty) \\ u'(t) + Cu(t) - B^*x'(t) = 0, & t \in (0, +\infty) \\ x(0) = x_0, x'(0) = y_0, u(0) = u_0. \end{cases}$$

The corresponding Cauchy problem can be written as

$$(11) \quad z'(t) = \mathcal{A}_c z(t), z(0) = z_0 \in \mathcal{D}(\mathcal{A}_c).$$

Recall that the system (11) is the system (3) with $\widehat{C} = -C$ and that $(\mathcal{A}_c, \mathcal{D}(\mathcal{A}_c))$ is given by

$$\mathcal{A}_c z = (y, -Ax - Bu, B^*y + Cu), \forall z = (x, y, u) \in \mathcal{D}(\mathcal{A}_c),$$

with

$$\mathcal{D}(\mathcal{A}_c) = \{(x, y, u) \in V \times V \times D(C), Ax + Bu \in X\}.$$

Note also that Proposition 2.2 still holds. In order to get uniform stability we will need the following assumptions:

(O) (Observability inequality) There exists a time $T > 0$ and a constant $c(T) > 0$ (which only depends on T) such that, for all $z_0 \in \mathcal{D}(\mathcal{A}_c)$, the solution $z_1(t) = (x_1(t), y_1(t), u_1(t))$ of (11) satisfies the following observability estimate:

$$(12) \quad \int_0^T \|D^* u_1(s)\|_W^2 ds \geq c(T) \|z_0\|_{\mathcal{H}}^2.$$

(H) (Transfer function estimate) Assume that for every $\lambda \in \mathbb{C}_+ = \{\lambda \in \mathbb{C} | \Re \lambda > 0\}$

$$\lambda \in \mathbb{C}_+ \rightarrow H(\lambda) = -D^*(\lambda I + C + \lambda B^*(\lambda^2 + A)^{-1}B)^{-1}D \in \mathcal{L}(W),$$

is bounded on $C_\beta = \{\lambda \in \mathbb{C} | \Re \lambda = \beta\}$, where β is a positive constant.

Theorem 4.1. *Assume that assumption (H) is satisfied or that $D \in \mathcal{L}(U)$. Then system (3) is exponentially stable, which means that the energy of the system satisfies*

$$(13) \quad E(t) \leq c e^{-\omega t} E(0), \forall t \in [0, +\infty),$$

where c and ω are two positive constants independent of the initial data $z_0 \in \mathcal{D}(\mathcal{A}_d)$ if and only if the inequality (12) is satisfied.

By using [6, Theorem 5.1] and [2, Proposition 2.1] we have the following characterization of the uniform stability of (3) by a frequency criteria (Hautus test).

Corollary 4.2. *Assume that assumption (H) is satisfied or that $D \in \mathcal{L}(U)$. Then system (3) is exponentially stable in the energy space if and only if there exists a constant $\delta > 0$ such that for all $w \in \mathbb{R}, z \in \mathcal{D}(\mathcal{A}_c)$ we have*

$$(14) \quad \|(iw - \mathcal{A}_c)z\|_{\mathcal{H}}^2 + \left\| \begin{pmatrix} 0 & 0 & D^* \end{pmatrix} z \right\|_U^2 \geq \delta \|z\|_{\mathcal{H}}^2.$$

Proof. (of Theorem 4.1). Let $z(t) = (x(t), y(t), u(t))$ be the solution of (3) with initial datum $z_0 \in \mathcal{D}(\mathcal{A}_d)$. Consider $z_1(t) = (x_1(t), y_1(t), u_1(t))$ the solution of (11) with initial datum $z_0 \in \mathcal{D}(\mathcal{A}_d)$. Let $z_2(t) = (x_2(t), y_2(t), u_2(t))$ be such that $z_2(t) = z(t) - z_1(t)$. Then z_2 is solution of (8) and due to Lemma 3.1 its last component u_2 satisfies (9) if $D \in \mathcal{L}(U)$. Otherwise, (9) holds true due to assumption (H). Since $u = u_1 + u_2$, we get

$$\begin{aligned} \|z_0\|_{\mathcal{H}}^2 &\lesssim \|D^* u_1\|_{L^2(0,T;W)}^2 && \text{estimate (12)} \\ &\lesssim \|D^* u\|_{L^2(0,T;W)}^2 + \|D^* u_2\|_{L^2(0,T;W)}^2 && \text{(triangle inequality)} \\ &\lesssim \|D^* u\|_{L^2(0,T;W)}^2 && \text{(estimate (9)).} \end{aligned}$$

Indeed x_2, u_2 satisfies the system

$$(15) \quad \begin{cases} x_2''(t) + Ax_2(t) + Bu_2(t) = 0, & t \in (0, +\infty) \\ u_2'(t) + Cu_2(t) - B^* x_2'(t) = -DD^* u(t), & t \in (0, +\infty) \\ x_2(0) = 0, x_2'(0) = 0, u_2(0) = 0. \end{cases}$$

Extend $D^* u$ by zero on $\mathbb{R} \setminus [0, T]$. Since the system (15) is reversible by time we solve the system on \mathbb{R} . We obtain a function $z \in C(\mathbb{R}; V) \cap C^1(\mathbb{R}; V) \cap L^2(\mathbb{R}; V)$ which is null for all $t \leq 0$. Let $\widehat{x}_2(\lambda)$, and $\widehat{u}_2(\lambda)$, where

$\lambda = \gamma + i\eta$, $\Re(\lambda) = \gamma > 0$ and $\eta \in \mathbb{R}$, be the respective Laplace transforms of x_2 and u_2 with respect to t . Then \widehat{x}_2 and \widehat{u}_2 satisfy

$$(16) \quad \begin{cases} \lambda^2 \widehat{x}_2(\lambda) + A\widehat{x}_2(\lambda) + B\widehat{u}_2(\lambda) = 0, \\ \lambda \widehat{u}_2(\lambda) + C\widehat{u}_2(\lambda) - B^* \lambda \widehat{x}_2(\lambda) = -DD^* \widehat{u}(\lambda). \end{cases}$$

Since $\lambda^2 + A$ is invertible (Lax-Milgram lemma), we deduce from the first equation of the system (16) that

$$\widehat{x}_2 = -(\lambda^2 + A)^{-1} B \widehat{u}_2.$$

Substituting \widehat{x}_2 in the second equation of system (16), we get

$$(\lambda I + C + \lambda B^* (\lambda^2 + A)^{-1} B) \widehat{u}_2 = -DD^* \widehat{u}.$$

Noting that the invertibility of $\lambda I + C + \lambda B^* (\lambda^2 + A)^{-1} B$ follows from the invertibility of $\lambda I - \mathcal{A}_c$ we obtain

$$D^* \widehat{u}_2 = -[D^* (\lambda I + C + \lambda B^* (\lambda^2 + A)^{-1} B)^{-1} D] D^* \widehat{u}$$

and by assumption **(H)** estimate (9) holds. Finally, the inequality, $\|z_0\|_{\mathcal{H}}^2 \lesssim \|D^* u\|_{L^2(0,T;U)}^2$, implies that there is a constant $c_1(T)$ which depends only of T such that

$$E(0) - E(T) \geq c_1(T) E(0).$$

But it is well known (see for instance [4]) that the previous estimate is equivalent to (13). \square

5 Weaker decay

In the case of non exponential decay in the energy space we give sufficient conditions for weaker decay properties. The statement of our second result requires some notations.

Let $\mathcal{H}_1, \mathcal{H}_2$ be two Banach spaces such that

$$\mathcal{D}(\mathcal{A}_d) \subset \mathcal{H}_1 \subset \mathcal{H} \subset \mathcal{H}_2,$$

where

$$\|\cdot\|_{\mathcal{D}(\mathcal{A}_d)} \sim \|\cdot\|_{\mathcal{H}_1}$$

and

$$(17) \quad [\mathcal{H}_1; \mathcal{H}_2]_{\theta} = \mathcal{H}$$

for a fixed $\theta \in]0; 1[$, where $[\cdot; \cdot]$ denotes the interpolation space (see for instance [8]).

Let $G : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be such that G is continuous, invertible, increasing on \mathbb{R}_+ and suppose that the function $x \longmapsto \frac{1}{x^{1-\theta}} G(x)$ is increasing on $(0; 1)$.

Theorem 5.1. *Assume that the function G satisfies the above assumptions and that assumption **(H)** is satisfied or that $D \in \mathcal{L}(U)$. Then the following assertions hold true:*

1. *If for all non zero $z_0 \in \mathcal{D}(\mathcal{A}_d)$, the solution $z_1(t) = (x_1(t), y_1(t), u_1(t))$ of (11) satisfies the following observability estimate:*

$$(18) \quad \int_0^T \|D^* u_1(s)\|_U^2 ds \geq c(T) \|z_0\|_{\mathcal{H}}^2 G\left(\frac{\|z_0\|_{\mathcal{H}_2}^2}{\|z_0\|_{\mathcal{H}}^2}\right),$$

then we have

$$(19) \quad E(t) \lesssim \left[G^{-1}\left(\frac{1}{1+t}\right) \right]^{\frac{\theta}{1-\theta}} \|z_0\|_{\mathcal{D}(\mathcal{A}_d)}^2.$$

2. If for all non zero $z_0 \in \mathcal{D}(\mathcal{A}_d)$, the solution $z_1(t) = (x_1(t), y_1(t), u_1(t))$ of (11) satisfies the following observability estimate:

$$(20) \quad \int_0^T \|D^* u_1(s)\|_U^2 ds \geq c(T) \|z_0\|_{\mathcal{H}_2}^2,$$

then we have

$$(21) \quad E(t) \lesssim \frac{1}{(1+t)^{\frac{\theta}{1-\theta}}} \|z_0\|_{\mathcal{D}(\mathcal{A}_d)}^2.$$

Proof. 1. Using the same arguments as in the proof of Theroem 4.1 we get from (18)

$$\forall z_0 \in \mathcal{D}(\mathcal{A}_d), \int_0^T \|D^* u(s)\|_U^2 ds \geq c(T) \|z_0\|_{\mathcal{H}}^2 G\left(\frac{\|z_0\|_{\mathcal{H}_2}^2}{\|z_0\|_{\mathcal{H}}^2}\right).$$

The sequel follows the proof of Theorem 2.4 of [4], therefore we give the outlines below. Using (17) and the interpolation inequality

$$\|z_0\|_{\mathcal{H}} \leq \|z_0\|_{\mathcal{H}_1}^{1-\theta} \|z_0\|_{\mathcal{H}_2}^{\theta}$$

we easily check

$$\frac{\|z_0\|_{\mathcal{H}_2}^2}{\|z_0\|_{\mathcal{H}}^2} \geq \frac{\|z_0\|_{\mathcal{H}}^{\frac{2-2\theta}{\theta}}}{\|z_0\|_{\mathcal{H}_1}^{\frac{2-2\theta}{\theta}}}, \quad \forall z_0 \in \mathcal{D}(\mathcal{A}_d).$$

Consequently, using (7) and the fact that the function $t \mapsto \|z(t)\|_{\mathcal{H}}$ is nonincreasing and G is increasing we obtain the existence of a constant $K_1 > 0$ such that

$$\|z(T)\|_{\mathcal{H}}^2 \leq \|z(0)\|_{\mathcal{H}}^2 - K_1 \|z(0)\|_{\mathcal{H}}^2 G\left(\frac{\|z(T)\|_{\mathcal{H}}^{\frac{2-2\theta}{\theta}}}{\|z(0)\|_{\mathcal{H}_1}^{\frac{2-2\theta}{\theta}}}\right).$$

Applying the same arguments on successive intervals $[kT, (k+1)T]$, $k = 1, 2, \dots$ we obtain the existence of a constant K_2 such that

$$\|z((k+1)T)\|_{\mathcal{H}}^2 \leq \|z(kT)\|_{\mathcal{H}}^2 - K_2 \|z(kT)\|_{\mathcal{H}}^2 G\left(\frac{\|z((k+1)T)\|_{\mathcal{H}}^{\frac{2-2\theta}{\theta}}}{\|z(0)\|_{\mathcal{H}_1}^{\frac{2-2\theta}{\theta}}}\right), \quad \forall z_0 \in \mathcal{D}(\mathcal{A}_d).$$

If we set $\mathcal{E}_k = G\left(\frac{\|z(kT)\|_{\mathcal{H}}^{\frac{2-2\theta}{\theta}}}{\|z(0)\|_{\mathcal{H}_1}^{\frac{2-2\theta}{\theta}}}\right)$, the previous inequality, the property of G and the fact that $t \mapsto \|z(t)\|_{\mathcal{H}}$ is nonincreasing then we get

$$\frac{\|z((k+1)T)\|_{\mathcal{H}}^2}{\|z(kT)\|_{\mathcal{H}}^2} \frac{\mathcal{E}_k}{\mathcal{E}_{k+1}} \mathcal{E}_k \leq \mathcal{E}_k - K_2 \mathcal{E}_{k+1}^2.$$

Equivalently, we have

$$(22) \quad \frac{\frac{1}{\left[\frac{\|z(kT)\|_{\mathcal{H}}^{\frac{2-2\theta}{\theta}}}{\|z(0)\|_{\mathcal{H}_1}^{\frac{2-2\theta}{\theta}}}\right]^{\frac{\theta}{1-\theta}}} G\left(\frac{\|z(kT)\|_{\mathcal{H}}^{\frac{2-2\theta}{\theta}}}{\|z(0)\|_{\mathcal{H}_1}^{\frac{2-2\theta}{\theta}}}\right)}{\frac{1}{\left[\frac{\|z((k+1)T)\|_{\mathcal{H}}^{\frac{2-2\theta}{\theta}}}{\|z(0)\|_{\mathcal{H}_1}^{\frac{2-2\theta}{\theta}}}\right]^{\frac{\theta}{1-\theta}}} G\left(\frac{\|z((k+1)T)\|_{\mathcal{H}}^{\frac{2-2\theta}{\theta}}}{\|z(0)\|_{\mathcal{H}_1}^{\frac{2-2\theta}{\theta}}}\right)} \mathcal{E}_{k+1} \leq \mathcal{E}_k - K_2 \mathcal{E}_{k+1}^2.$$

Combining (22) and the fact that the function $x \mapsto \frac{1}{x^{\frac{\theta}{1-\theta}}} G(x)$ is increasing on $(0; 1)$, we get

$$\mathcal{E}_{k+1} \leq \mathcal{E}_k - K_2 \mathcal{E}_{k+1}^2.$$

We thus deduce the existence of a constant $M > 0$ such that $\mathcal{E}_k \leq \frac{M}{k+1}$ and we finally get (19).

2. As for 1. the proof is similar to the second assertion of Theorem 2.4 of [4] which is based on Lemma 5.2 of [3] and is left to the reader. □

6 Examples

Beam System

We consider the following beam equation:

$$(23) \quad \begin{cases} u_{tt}(x, t) + u^{(4)}(x, t) = 0, & 0 < x < 1, t \in [0, \infty) \\ \eta_t(t) + \beta \eta(t) - u_t(1, t) = 0, & 0 < x < 1, t \in [0, \infty) \\ u(0, t) = u'(0, t) = u''(1, t) = 0, & t \in [0, \infty) \\ u'''(1, t) = \eta(t) \end{cases}$$

with the initial conditions

$$u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), \eta(0) = \eta_0.$$

In this case

$$\begin{aligned} X &= L^2(0, 1), U = \mathbb{C}, V = \{u \in H^2(0, 1) : u(0) = u'(0) = 0\}, \\ \mathcal{D}(A) &= \{u \in H^4(0, 1) : u(0) = u'(0) = u''(1) = 0, u^{(3)}(1) = 0\}, \\ a(u, v) &= \int_0^1 \bar{u}^{(2)} v^{(2)} dx \quad (u, v \in V), \quad Au = u^{(4)} \quad (u \in \mathcal{D}(A)), \\ B^* &= \delta_1, B^* \varphi = \varphi(1) \quad (\varphi \in V). \\ (B\eta, \varphi)_{V', V} &= \bar{\eta} \varphi(1) \quad (\eta \in \mathbb{C}, \varphi \in V), \end{aligned}$$

and

$$\begin{aligned} \widehat{C} : \quad \mathbb{C} &\rightarrow \mathbb{C} \\ \eta &\rightarrow -\beta \eta \quad , \end{aligned}$$

where β is a positive constant.

Note that \widehat{C} is bounded, so we only need to find the observability inequality in order to deduce the type of stability of the system. Since $B \in \mathcal{L}(U, V')$ then $B\eta = \eta \cdot B1$, and since $B1 \in V'$ and $A \in \mathcal{L}(V, V')$ then there exists a unique $u_0 \in V$ such that $B1 = Au_0$. Indeed, it is easy to check that $u_0(x) = \frac{x^2}{2} - \frac{x^3}{6}$. Moreover, $C = 0$ and $D = \sqrt{\beta}$.

Remark that in this case

$$\mathcal{D}(\mathcal{A}_c) = \mathcal{D}(\mathcal{A}_d) = \{(u, v, \eta) \in V \times V \times \mathbb{C} : Au + B\eta \in L^2(0, 1)\},$$

and

$$(24) \quad \mathcal{A}_c \begin{pmatrix} u \\ v \\ \eta \end{pmatrix} = \begin{pmatrix} v \\ -Au - B\eta \\ B^*v \end{pmatrix}.$$

Note that since $\mathcal{D}(\mathcal{A}_c)$ is compactly injected in \mathcal{H} , then \mathcal{A}_c has a compact resolvent and thus its spectrum is discrete. In addition, since \mathcal{A}_c is a skew-adjoint real operator, then its spectrum is constituted of pure imaginary conjugate eigenvalues. Now, let $\lambda = i\mu \in \sigma(\mathcal{A}_c)$ with U_μ an associated eigenvector then $\bar{\lambda} = -i\mu \in \sigma(\mathcal{A}_c)$ with \bar{U}_μ an associated eigenvector. Since the eigenvalues are conjugates, it is sufficient then to study $\mu \geq 0$.

Lemma 6.1. *The eigenvalues of \mathcal{A}_c are algebraically simple. Moreover, $0 \in \sigma(\mathcal{A}_c)$ and for every $\lambda = i\mu \in \sigma(\mathcal{A}_c)$, $\mu > 0$, μ satisfies the following characteristic equation,*

$$(25) \quad f(\mu) = \mu\sqrt{\mu} + \mu\sqrt{\mu} \cosh(\sqrt{\mu}) \cos(\sqrt{\mu}) + \sin(\sqrt{\mu}) \cosh(\sqrt{\mu}) - \cos(\sqrt{\mu}) \sinh(\sqrt{\mu}) = 0.$$

Proof. First it is easy to see that 0 is a simple eigenvalue of \mathcal{A}_c and that an associated eigenvector is $U = \eta(-u_0, 0, 1)^\top, \eta \in \mathbb{C}$.

Let $\lambda = i\mu \in \sigma(\mathcal{A}_c)$, $\mu > 0$, and let $U = (u, v, \eta)^\top \in \mathcal{D}(\mathcal{A}_c)$ be a nonzero associated eigenvector. Then U satisfies

$$\mathcal{A}_c(u, v, \eta)^\top = \lambda(u, v, \eta)^\top,$$

which is equivalent to

$$(26) \quad \begin{cases} v = \lambda u \\ B^*v = \lambda \eta \\ -Au - \eta Au_0 = \lambda v = \lambda^2 u. \end{cases}$$

We then deduce that

$$A(u + \eta u_0) = -\lambda^2 u, B^*u = \lambda u(1) = \eta.$$

But as $U \in \mathcal{D}(\mathcal{A}_c)$, then $Au + B\eta = A(u + \eta u_0) \in L^2(0, 1)$, which implies that $u + \eta u_0 \in \mathcal{D}(A)$ and that $u \in H^4(0, 1)$ satisfies

$$(27) \quad u(0) = u'(0) = u''(1) = 0, u'''(1) = \eta.$$

However, $A(u + \eta u_0) = (u + \eta u_0)^{(4)} = u^{(4)}$, thus we need to solve $u^{(4)} = -\lambda^2 u = \mu^2 u$, $u(1) = \eta$ with u satisfying (27). We deduce that u could be written as

$$u = c_1 \sin(\sqrt{\mu}x) + c_2 \sinh(\sqrt{\mu}x) + c_3 \cos(\sqrt{\mu}x) + c_4 \cosh(\sqrt{\mu}x),$$

with $C = (c_1, c_2, c_3, c_4)^\top$ satisfying

$$(28) \quad \widetilde{M}C = V_0$$

where

$$\widetilde{M} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ -\sin(\sqrt{\mu}) & \sinh(\sqrt{\mu}) & -\cos(\sqrt{\mu}) & \cosh(\sqrt{\mu}) \\ -\cos(\sqrt{\mu}) & \cosh(\sqrt{\mu}) & \sin(\sqrt{\mu}) & \sinh(\sqrt{\mu}) \end{pmatrix}, V_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{\eta}{\mu\sqrt{\mu}} \end{pmatrix}.$$

We first remark that $\eta \neq 0$. Otherwise, since u satisfies $u^{(4)} = \mu^2 u$ and the boundary conditions $u(1) = u''(1) = u'''(1) = 0$, then there exists a constant $c \in \mathbb{R}$ such that u is given by

$$u(x) = c(\sinh(\sqrt{\mu}(1-x)) + \sin(\sqrt{\mu}(1-x))).$$

But $\cosh(\sqrt{\mu}) + \cos(\sqrt{\mu}) > 0$, then $u'(0) = 0$ implies that $c = 0$ and hence $U = (u, \lambda u, \eta)^\top = 0$ which is a contradiction.

Consequently, each eigenvalue of \mathcal{A}_c is simple. In fact, suppose to the contrary that there exists $\mu \neq 0$ such that $\lambda = i\mu$ is not algebraically simple. Then as \mathcal{A}_c is skew-adjoint, $\lambda = i\mu$ is not geometrically simple. Thus there exists at least two independent eigenvectors $U_i = (u_i, v_i, \eta_i)$, $i = 1, 2$, corresponding to λ , and hence $U = \eta_2 U_1 - \eta_1 U_2 = (u, v, \eta) = (u, v, 0)$ is an eigenvector which is impossible.

Going back to (28), we get from the first three equations,

$$c_2 = -c_1, \quad c_4 = -c_3, \quad c_3 = -c_1 \frac{\sin(\sqrt{\mu}) + \sinh(\sqrt{\mu})}{\cos(\sqrt{\mu}) + \cosh(\sqrt{\mu})}.$$

Therefore the last equation of (28) becomes

$$-\frac{2c_1(1 + \cos(\sqrt{\mu}) \cosh(\sqrt{\mu}))}{\cos(\sqrt{\mu}) + \cosh(\sqrt{\mu})} = \frac{\eta}{\mu\sqrt{\mu}}.$$

As $\eta \neq 0$ then the determinant of \widetilde{M} which is given by $\det(\widetilde{M}) = -2(1 + \cos(\sqrt{\mu}) \cosh(\sqrt{\mu}))$ is nonzero and C is given by

$$C = \widetilde{M}^{-1}V_0 = \frac{\eta}{2\mu\sqrt{\mu}(1 + \cos(\sqrt{\mu}) \cosh(\sqrt{\mu}))} \begin{pmatrix} -\cos(\sqrt{\mu}) - \cosh(\sqrt{\mu}) \\ \cos(\sqrt{\mu}) + \cosh(\sqrt{\mu}) \\ \sin(\sqrt{\mu}) + \sinh(\sqrt{\mu}) \\ -\sin(\sqrt{\mu}) - \sinh(\sqrt{\mu}) \end{pmatrix}.$$

Substituting C in the condition $u(1) = \eta$, we finally deduce that μ satisfies the characteristic equation (25). \square

Now, we study the asymptotic behavior of the eigenvalues of \mathcal{A}_c .

Lemma 6.2. *There exists $k_0 \in \mathbb{N}$ large enough such that for all $k \geq k_0$ there exists one and only one $\lambda_k = i\mu_k$ eigenvalue of \mathcal{A}_c with $\sqrt{\mu_k} \in [k\pi, (k+1)\pi]$. Moreover, as $k \rightarrow \infty$, we have the following*

$$(29) \quad \sqrt{\mu_k} = \frac{\pi}{2} + k\pi + \frac{1}{k^3\pi^3} + o\left(\frac{1}{k^3}\right).$$

Let $U_{1,k} = (u_{1,k}, \lambda_k u_{1,k}, \eta_{1,k})$ be the associated normalized eigenvector. Then,

$$(30) \quad |\eta_{1,k}|^2 = \frac{4}{k^4} + o\left(\frac{1}{k^4}\right).$$

Proof. First step. Let $z = \sqrt{\mu}$ where $i\mu \in \sigma(\mathcal{A}_c)$ and $\mu > 0$. Then by (25), we have

$$f(z^2) = z^3 + \cosh z(z^3 \cos z + \sin z) - \cos z \sinh z = 0.$$

Replacing $\cosh z = \frac{e^z + e^{-z}}{2}$ and $\sinh z = \frac{e^z - e^{-z}}{2}$ in $f(z^2)$ and dividing by $\frac{z^3 e^z}{2}$, we deduce that z satisfies $\tilde{f}(z) = 0$, where

$$\tilde{f}(z) = \cos z + \frac{\sin z - \cos z}{z^3} + 2e^{-z} + e^{-2z} \left(\cos z + \frac{\cos z}{z^3} + \frac{\sin z}{z^3} \right).$$

For z large enough we have

$$\tilde{f}(z) = \cos z + O(1/z^3).$$

It can be easily checked that for k large enough, \tilde{f} doesn't admit any root outside the ball $B_k = B(z_k^0, \frac{1}{k^2})$, with $z_k^0 = \frac{\pi}{2} + k\pi$. Then by Rouché's Theorem applied on B_k , we deduce that for k large enough there exists a unique root z_k of \tilde{f} in $[k\pi, (k+1)\pi]$. Moreover, z_k satisfies

$$z_k = \frac{\pi}{2} + k\pi + \epsilon_k,$$

with $\epsilon_k = o(1)$. Since z_k satisfies $\tilde{f}(z_k) = 0$, then ϵ_k satisfies

$$\cos\left(\frac{\pi}{2} + k\pi + \epsilon_k\right) + \frac{\sin\left(\frac{\pi}{2} + k\pi + \epsilon_k\right) + o(1)}{k^3\pi^3 + o(k^3)} + O(e^{-z_k}) = 0.$$

Hence

$$-\sin(\epsilon_k) + \frac{\cos(\epsilon_k)}{k^3\pi^3} + o\left(\frac{1}{k^3}\right) = 0,$$

and thus

$$-k^3\epsilon_k + o(k^3\epsilon_k^2) + \frac{1}{\pi^3} + o(k^2\epsilon_k) + o(1) = 0,$$

which gives

$$\epsilon_k = \frac{1}{\pi^3 k^3} + o(1/k^3).$$

Therefore, (29) follows for $\mu_k = z_k^2$.

Second step. Set $\beta_k = \frac{\sin(z_k) + \sinh(z_k)}{\cos(z_k) + \cosh(z_k)}$. Then

$$(31) \quad \beta_k = \frac{e^{z_k} + 2\sin(z_k) - e^{-z_k}}{e^{z_k} + 2\cos(z_k) + e^{-z_k}} = 1 + o(e^{-z_k}).$$

By the proof of Lemma 6.1, the last component η_k^1 of U_k^1 is nonzero and thus

$$U_k = (u_k, iz_k^2 u_k, 1) = \frac{1}{\eta_{1,k}} U_{1,k}$$

is an associated eigenvector to iz_k^2 with u_k having the form,

$$u_k(x) = c_{1k} \sin(z_k x) + c_{2k} \sinh(z_k x) + c_{3k} \cos(z_k x) + c_{4k} \cosh(z_k x),$$

with

$$c_{2k} = -c_{1k}, \quad c_{4k} = -c_{3k}, \quad c_{3k} = -\beta_k c_{1k}.$$

It follows that

$$(32) \quad u_k(x) = c_{1k} [(\sin(z_k x) - \sinh(z_k x) - \cos(z_k x) + \cosh(z_k x)) + (\beta_k - 1)(-\cos(z_k x) + \cosh(z_k x))].$$

In order to get the behavior of $\eta_k = \frac{1}{\|U_k\|}$, it is enough to compute the integral $\int_0^1 |u_k|^2 dx$. Indeed, multiplying $u_k^{(4)} = -\lambda^2 u_k = \mu^2 u_k$ by \bar{u}_k , integrating by parts and noting that $u_k(0) = u_k'(0) = 0$, $u_k(1) = u_k'''(1) = 1$ we obtain

$$\int_0^1 |u_k''|^2 dx = \mu_k^2 \int_0^1 u_k^2 dx - 1,$$

and hence

$$\|U_k\|^2 = \int_0^1 u_{kxx}^2 dx + \mu_k^2 \int_0^1 u_k^2 dx + 1 = 2\mu_k^2 \int_0^1 u_k^2 dx.$$

Since

$$\begin{aligned} 2z_k^3 c_{1k} &= \frac{-\cos z_k - \frac{e^{z_k}}{2}(1 + e^{-2z_k})}{1 + \frac{e^{z_k}}{2} \cos z_k (1 + e^{-2z_k})} \\ &= \frac{-1 + O(e^{-k})}{(-1)^{k+1} \sin \epsilon_k + O(e^{-k})}, \end{aligned}$$

we deduce that

$$(33) \quad c_{1k} = \frac{(-1)^k}{2} + o(1).$$

As

$$\int_0^1 (\sin(zx) - \sinh(zx) - \cos(zx) + \cosh(zx))^2 dx = \int_0^1 (\sin(zx) - \cos(zx))^2 dx + o(1) = 1 + o(1),$$

and

$$\int_0^1 (-\cos(zx) + \cosh(zx))^2 dx = \frac{e^{2z}}{8z} + o\left(\frac{e^{2z}}{8z}\right),$$

we consequently deduce due to (31), (32) and (33) that

$$\int_0^1 u_k^2(x) dx = \frac{1}{4} + o(1), \quad \text{and} \quad \|U_k\|^2 = \frac{k^4}{4} + o(k^4).$$

Hence (30) holds. □

Proposition 6.3. *Let $U_1 = (u_1, v_1, \eta_1)^T$ be the solution of the conservative problem (24) with initial datum $U_0 \in \mathcal{D}(\mathcal{A}_c)$. Then there exists $T > 0$ and $c > 0$ depending on T such that*

$$(34) \quad \int_0^T |\eta_1(t)|^2 dt \geq c \|U_0\|_{D(A^{-1})}^2.$$

Proof. We arrange the elements of $\sigma(\mathcal{A}_c)$ in increasing order.

Let $J = \{i\mu : |\mu| < \mu_{k_0}\}$. Then $\sigma(\mathcal{A}_c) = J \cup \{i\mu_k : |k| \geq k_0\}$ and $(U_\mu)_{\mu \in J} \cup (U_{1,k})_{|k| \geq k_0}$ forms a Hilbert basis of \mathcal{H} . We may write

$$U_0 = \sum_{\mu \in J} u_0^\mu U_\mu + \sum_{|k| \geq k_0} u_0^{(k)} U_{1,k}.$$

Moreover,

$$\eta_1(t) = \sum_{\mu \in J} u_0^\mu e^{i\mu t} \eta_\mu + \sum_{|k| \geq k_0} u_0^{(k)} e^{i\mu_k t} \eta_{1,k}.$$

Note that $\mu_{k+1} - \mu_k \geq \frac{\pi}{2}$ for $|k| \geq k_0$. Set $\gamma_0 = \min\{\frac{\pi}{2}, \min\{|\mu - \mu'| : \mu \in J, \mu' \in J\}\}$. As $|\mu - \mu'| \geq \gamma_0 > 0$ for all consecutive $\mu \in \sigma(\mathcal{A}_c), \mu' \in \sigma(\mathcal{A}_c)$. Then using Ingham's inequality there exists $T > 2\pi\gamma_0 > 0$ and a constant c depending on T such that

$$\int_0^T |\eta_1(t)|^2 dt \geq c \left(\sum_{\mu \in J} |u_0^\mu \eta_\mu|^2 + \sum_{|k| \geq k_0} |u_0^{(k)} \eta_{1,k}|^2 \right).$$

Due to Lemma 6.2, we have that $|\eta_{1,k}|^2 \sim \frac{1}{k^4}$. we deduce using Ingham's inequality the existence of $T > 0$ such that

$$(35) \quad \int_0^T |\eta_1|^2 dt \gtrsim \sum_{\mu \in J} |u_0^\mu|^2 |\mu|^{-2} + \sum_{|k| \geq k_0} \frac{|u_0^{(k)}|^2}{k^4}.$$

Therefore, we obtain (34) as required. \square

Theorem 6.4. *Let $U_0 \in \mathcal{D}(\mathcal{A}_d)$ and let U be the solution of the corresponding dissipative problem*

$$U_t = \mathcal{A}_d U, \quad U(0) = U_0 \in \mathcal{D}(\mathcal{A}_d).$$

Then U satisfies,

$$(36) \quad \|U(t)\|^2 \lesssim \frac{1}{1+t} \|U_0\|_{\mathcal{D}(\mathcal{A}_d)}^2.$$

Proof. Since the operator $D \in \mathcal{L}(U)$, then Lemma 3.1 holds true.

Set $\mathcal{H}_1 = \mathcal{D}(\mathcal{A}_c)$ and $\mathcal{H}_2 = \mathcal{D}(\mathcal{A}_c^{-1})$, obtained by means of the inner product in X . Then $\mathcal{H} = [\mathcal{H}_1; \mathcal{H}_2]_{1/2}$. By Proposition 6.3, we have

$$\int_0^T \|D^* u_1(s)\|_U^2 ds \geq c_T \|u_0\|_{\mathcal{H}_2}^2.$$

By Theorem 5.1 applied for $\theta = 1/2$, we therefore obtain (36). \square

Example on uniform stability

Consider the following system,

$$(37) \quad \begin{cases} u_{tt}(x, t) + u^{(4)}(x, t) + \alpha \theta_{xx}(x, t) = 0, & t \in [0, \infty), 0 < x < 1 \\ \theta_t(x, t) + \beta \theta(x, t) - \alpha u_{txx}(x, t) = 0, & t \in [0, \infty), 0 < x < 1 \\ u(0, t) = u(1, t) = u''(0, t) = u''(1, t) = 0 & t \in [0, \infty) \\ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), \theta(x, 0) = \theta_0(x), & 0 < x < 1 \end{cases}$$

with $\alpha > 0, \beta > 0$. Define the following spaces,

$$V = H^2(0, 1) \cap H_0^1(0, 1), X = U = L^2(0, 1),$$

and the following operators,

$$\mathcal{D}(A) = \{u \in H^4(0, 1) \cap H_0^1(0, 1) : u_{xx}(0) = u_{xx}(1) = 0\}, Au = u_{xxxx} \in L^2(0, 1),$$

$$\begin{aligned} \widehat{C} : L^2(0, 1) &\rightarrow L^2(0, 1) \\ \theta &\rightarrow \beta\theta. \end{aligned}$$

Remark that \widehat{C} is a bounded operator on $L^2(0, 1)$. Moreover, B and B^* are given by

$$\begin{aligned} B : U &\rightarrow V' & , \quad B^* : V &\rightarrow U \\ \theta &\rightarrow \alpha\theta_{xx} & u &\rightarrow \alpha u_{xx}, \end{aligned}$$

and $D, D^* \in \mathcal{L}(U)$ with $D\theta = D^*\theta = \sqrt{\beta}\theta$. The norm defined on the energy space $\mathcal{H} = V \times X \times U$ is given by

$$\|(u, v, \theta)^\top\|_{\mathcal{H}}^2 = \int_0^1 |u_{xx}|^2 dx + \int_0^1 |v|^2 dx + \int_0^1 |\theta|^2 dx$$

We moreover have

$$\mathcal{D}(\mathcal{A}_d) = \mathcal{D}(\mathcal{A}_c) = \{(u, v, \theta)^\top \in V \times V \times U : u^{(4)} + \theta_{xx} \in L^2(\Omega)\}.$$

The associated conservative system is given by

$$(38) \quad \begin{cases} u_{tt}(x, t) + u^{(4)}(x, t) + \alpha\theta_{xx}(x, t) = 0, & t \in [0, \infty), 0 < x < 1 \\ \theta_t(x, t) - \alpha u_{txx}(x, t) = 0, & t \in [0, \infty), 0 < x < 1 \\ u(0, t) = u(1, t) = u''(0, t) = u''(1, t) = 0 & t \in [0, \infty) \\ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), \theta(x, 0) = \theta_0(x), & 0 < x < 1. \end{cases}$$

In the following proposition we prove that the solution u, θ of (38) satisfies the required observability inequality (assumption (O)), which is enough to deduce the exponential stability of (37) as $D \in \mathcal{L}(U)$.

Proposition 6.5. *Let $U_0 = (u_0, u_1, \theta_0)^\top \in \mathcal{H}$. Then the solution (u, θ) of (38) satisfies*

$$(39) \quad \int_0^T |\theta(t)|^2 dt \gtrsim \|U_0\|_{\mathcal{H}}^2.$$

Proof. Writing $(u_0, u_1, \theta_0)^\top \in \mathcal{D}(\mathcal{A}_c)$ with respect to the basis $(\sin(k\pi x))_{k \in \mathbb{N}^*}$ of $L^2(0, 1)$, we have

$$u_0 = \sum_{k \in \mathbb{N}^*} u_k^0 \sin(k\pi x), u_1 = \sum_{k \in \mathbb{N}^*} u_k^1 \sin(k\pi x), \theta_0 = \sum_{k \in \mathbb{N}^*} \theta_k^0 \sin(k\pi x).$$

The solution (u, θ) of (38) is thus given by

$$u(t) = \sum_{k \in \mathbb{N}^*} u_k(t) \sin(k\pi x) \quad \text{and} \quad \theta(t) = \sum_{k \in \mathbb{N}^*} \theta_k(t) \sin(k\pi x).$$

By the second equation of (38),

$$\theta'_k(t) + \alpha k^2 \pi^2 u'_k(t) = 0, \forall k \in \mathbb{N}^*.$$

Due to the initial conditions we deduce that

$$\theta_k(t) = -\alpha k^2 \pi^2 u_k(t) + \theta_k^0 + \alpha k^2 \pi^2 u_k^0.$$

Replacing u and θ in the first equation of (38), we deduce that

$$u_k''(t) + k^4\pi^4(1 + \alpha^2)u_k(t) = \alpha k^2\pi^2(\theta_k^0 + \alpha k^2\pi^2 u_k^0), \forall k \in \mathbb{N}^*,$$

hence

$$u_k(t) = \frac{\alpha(\theta_k^0 + \alpha k^2\pi^2 u_k^0)}{k^2\pi^2(1 + \alpha^2)} + c_1 \cos(k^2\pi^2\sqrt{1 + \alpha^2}t) + c_2 \sin(k^2\pi^2\sqrt{1 + \alpha^2}t),$$

where

$$c_1 = \frac{-\alpha\theta_k^0 + k^2\pi^2 u_k^0}{k^2\pi^2(1 + \alpha^2)}, c_2 = \frac{u_k^1}{k^2\pi^2\sqrt{1 + \alpha^2}},$$

obtained by the initial conditions $u_k(0) = u_k^0$, $u_k'(0) = u_k^1$ and $\theta_k(0) = \theta_k^0$.

Finally,

$$\begin{aligned} \theta_k(t) = \frac{1}{(1 + \alpha^2)^{\frac{3}{2}}} & [\sqrt{1 + \alpha^2}(\theta_k^0 + \alpha k^2\pi^2 u_k^0) + \alpha\sqrt{1 + \alpha^2}(\alpha\theta_k^0 - k^2\pi^2 u_k^0) \cos(\sqrt{1 + \alpha^2}k^2\pi^2 t) \\ & - \alpha(1 + \alpha^2)u_k^1 \sin(\sqrt{1 + \alpha^2}k^2\pi^2 t)]. \end{aligned}$$

Set $T = \frac{2}{\sqrt{1 + \alpha^2}\pi}$. Then,

$$\begin{aligned} |\theta_k(t)|^2 &= \frac{1}{(1 + \alpha^2)^{\frac{5}{2}}\pi} [(2 + \alpha^4)(\theta_k^0)^2 - 2\alpha(-2 + \alpha^2)k^2\pi^2\theta_k^0 u_k^0 + \alpha^2(3k^4\pi^4(u_k^0)^2 + (1 + \alpha^2)(u_k^1)^2)] \\ &= \frac{\alpha^2(1 + \alpha^2)(u_k^1)^2}{(1 + \alpha^2)^{\frac{5}{2}}\pi} + (k^2 u_k^0 \quad \theta_k^0) M \begin{pmatrix} k^2 u_k^0 \\ \theta_k^0 \end{pmatrix}, \end{aligned}$$

where M is a square matrix given by

$$\begin{pmatrix} \frac{3\alpha^2\pi^3}{(1 + \alpha^2)^{\frac{5}{2}}} & -\frac{\alpha(-2 + \alpha^2)\pi}{(1 + \alpha^2)^{\frac{5}{2}}} \\ -\frac{\alpha(-2 + \alpha^2)\pi}{(1 + \alpha^2)^{\frac{5}{2}}} & \frac{2 + \alpha^4}{\pi(1 + \alpha^2)^{\frac{5}{2}}} \end{pmatrix}.$$

But as

$$\det M = \frac{2\alpha^2\pi^2}{(1 + \alpha^2)^3} > 0, \text{ trace } M = \frac{2 + \alpha^4 + 3\alpha^2\pi^4}{\pi(1 + \alpha^2)^{\frac{5}{2}}} > 0,$$

we deduce that $\lambda_{\min} \geq c > 0$ (where λ_{\min} is the smallest eigenvalue of M) for some constant c independent of k and hence

$$\int_0^T |\theta_k(t)|^2 dt \geq T \left(\frac{\alpha^2(1 + \alpha^2)(u_k^1)^2}{(1 + \alpha^2)^{\frac{5}{2}}\pi} + \lambda_{\min}(M)(k^4(u_k^0)^2 + (\theta_k^0)^2) \right)$$

we get

$$\int_0^T |\theta(t)|^2 dt \gtrsim \sum_{k \in \mathbb{N}^*} (k^4(u_k^0)^2 + (\theta_k^0)^2 + (u_k^1)^2) \gtrsim \|U_0\|_{\mathcal{H}}^2.$$

We hence conclude (39) by denseness of $\mathcal{D}(\mathcal{A}_c)$ in \mathcal{H} . □

Recall that the energy of (u, θ) a solution of (38) is defined by

$$E(t) = \frac{1}{2} \left(\int_0^1 |u_{xx}|^2 dx + \int_0^1 |u_t|^2 dx + \int_0^1 |\theta|^2 dx \right).$$

Theorem 6.6. *Let $U_0 \in \mathcal{H}$. Then there exists $\omega > 0$ such that the energy of the solution (u, θ) of (38) satisfies*

$$(40) \quad E(t) \lesssim e^{-\omega t} E(0), \forall t \in [0, +\infty).$$

Proof. By Proposition 6.5, assumption (O) holds true. Then (40) follows by applying Theorem 4.1. □

Hybrid example-2D problem

Let Ω be a bounded domain of \mathbb{R}^2 whose boundary Γ satisfies

$$\Gamma = \Gamma_0 \cup \Gamma_1, \bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \phi, \text{ and } \text{meas } \Gamma_0 \neq 0.$$

We assume moreover that there exists a point $x_0 \in \mathbb{R}^2$ such that

$$\Gamma_0 = \{x \in \Gamma : m(x) \cdot \nu \leq 0\}, \Gamma_1 = \{x \in \Gamma : m(x) \cdot \nu \geq \omega > 0\},$$

for some constant $\omega > 0$, where $m(x) = x - x_0$ and $\nu = \nu(x)$ denotes the unit outward normal vector at $x \in \Gamma$. Denote by $R = \|m\|_\infty = \sup_{x \in \Omega} \|m(x)\|$.

Consider the following system,

$$(P_b) \quad \begin{cases} y_{tt}(x, t) - \Delta y(x, t) = 0, & x \in \Omega, t > 0, \\ y(x, t) = 0, & x \in \Gamma_0, t > 0, \\ ay_{tt}(x, t) + \partial_\nu y(x, t) + \eta(x, t) = 0, & x \in \Gamma_1, t > 0, \\ \eta_t(x, t) - y_t(x, t) + b\eta(x, t) = 0, & x \in \Gamma_1, t > 0, \\ y(x, 0) = y_0(x), y_t(x, 0) = y_1(x), & x \in \Omega, \\ \eta(x, 0) = \eta_0(x) & x \in \Gamma_1, \end{cases}$$

where a and b are two positive constants. In order to justify that the system could be written in the proposed general form, we introduce a proper functional setting. Let

$$X = L^2(\Omega) \times L^2(\Gamma_1)$$

endowed with the inner product,

$$\left((y, \xi)^\top, (\tilde{y}, \tilde{\xi})^\top \right)_X = \int_\Omega \langle y, \tilde{y} \rangle dx + \frac{1}{a} \int_{\Gamma_1} \langle \xi, \tilde{\xi} \rangle ds$$

and

$$W = \{y \in H^1(\Omega) : y = 0 \text{ on } \Gamma_0\} = H_{\Gamma_0}^1(\Omega), \quad U = L^2(\Gamma_1).$$

Define also V by

$$V = \{(y, \xi) \in W \times L^2(\Gamma_1) : ay = \xi \text{ on } \Gamma_1\},$$

and the operator $(A, \mathcal{D}(A))$ by

$$A(y, \xi)^\top = (-\Delta y, \partial_\nu y|_{\Gamma_1})^\top$$

with

$$\mathcal{D}(A) = \{x = (y, \xi)^\top \in V : y \in H^2(\Omega)\}.$$

We can easily check using Lax-Milgram lemma that $(A \pm iI)$ are surjective. In addition, since A is symmetric we deduce that A is self-adjoint. The corresponding form \tilde{a} is given by

$$\tilde{a}(u, \tilde{u}) = \int_\Omega \langle y_x, \tilde{y}_x \rangle dx, \quad u = (y, \xi)^\top \in V, \tilde{u} = (\tilde{y}, \tilde{\xi})^\top \in V.$$

In addition, we define for every $\eta \in U$ and $(y, \xi)^\top \in V$ the operators B and B^* by

$$B\eta = (0, \eta)^\top, \quad B^*(y, \xi)^\top = y|_{\Gamma_1}.$$

The operator $C = 0$ and the operator \hat{C} is given by

$$\hat{C}\eta = -b\eta, \quad \eta \in L^2(\Gamma_1).$$

Hence the system (P_b) can be written in the form of system (2).

Accordingly, we define the energy space

$$\mathcal{H} = V \times L^2(\Omega) \times L^2(\Gamma_1)^2,$$

endowed with the inner product

$$(u, \tilde{u})_{\mathcal{H}} = \int_{\Omega} \langle y_x, \tilde{y}_x \rangle dx + \int_{\Omega} \langle z, \tilde{z} \rangle dx + \frac{1}{a} \int_{\Gamma_1} \langle \xi, \tilde{\xi} \rangle ds + \int_{\Gamma_1} \langle \eta, \tilde{\eta} \rangle ds,$$

where $u = (y, \zeta, z, \xi, \eta)$, $\tilde{u} = (\tilde{y}, \tilde{\zeta}, \tilde{z}, \tilde{\xi}, \tilde{\eta}) \in \mathcal{H}$, and $\langle \cdot, \cdot \rangle$ represents the Hermitian product in \mathbb{C} . The associated norm will be denoted by $\|\cdot\|_{\mathcal{H}}$. Moreover, $(\mathcal{A}_d, \mathcal{D}(\mathcal{A}_d))$ is then given by

$$\mathcal{A}_d u = (z, \xi, \Delta y, -\partial_{\nu} y - \eta, z|_{\Gamma_1} - b\eta), \forall u = (y, \zeta, z, \xi, \eta) \in \mathcal{D}(\mathcal{A}_d),$$

with

$$\mathcal{D}(\mathcal{A}_d) = \{u = (y, \zeta, z, \xi, \eta) \in \mathcal{H} : y \in H^2(0, 1), z \in W, \zeta = ay|_{\Gamma_1} \xi = az|_{\Gamma_1}\}.$$

Hence, the previous problem (P_b) is formally equivalent to

$$(41) \quad u_t = \mathcal{A}_d u, \quad u(0) = u_0,$$

where $u_0 = (y_0, ay_0|_{\Gamma_1}, y_1, ay_1|_{\Gamma_1}, \eta_0)$. The energy of the system (P_b) is given by

$$E(t) = \frac{1}{2} \left(\int_{\Omega} |y_t|^2 dx + \int_{\Omega} |\nabla y|^2 dx + \frac{1}{a} \int_{\Gamma_1} |y_t|^2 ds + \int_{\Gamma_1} |\eta^2| ds \right),$$

and its derivative

$$\frac{d}{dt} E(t) = -b \int_{\Gamma_1} |\eta^2| ds.$$

The corresponding conservative system is defined by

$$(P_0) \quad \begin{cases} y_{tt}(x, t) - \Delta y(x, t) = 0, & x \in \Omega, t > 0, \\ y(x, t) = 0, & x \in \Gamma_0, t > 0, \\ ay_{tt}(x, t) + \partial_{\nu} y(x, t) + \eta(x, t) = 0, & x \in \Gamma_1, t > 0, \\ \eta_t(x, t) - y_t(x, t) = 0, & x \in \Gamma_1, t > 0, \\ y(x, 0) = y_0(x), y_t(x, 0) = y_1(x), & x \in \Omega, \\ \eta(x, 0) = \eta_0(x) & x \in \Gamma_1. \end{cases}$$

The initial value problem associated to the conservative system (P_0) is given by

$$(42) \quad u_t = \mathcal{A}_c u, \quad u(0) = u_0,$$

where

$$\mathcal{A}_c u = (z, \xi, \Delta y, -\partial_{\nu} y - \eta, z|_{\Gamma_1}), \forall u = (y, \zeta, z, \xi, \eta) \in \mathcal{D}(\mathcal{A}_c), \quad \mathcal{D}(\mathcal{A}_c) = \mathcal{D}(\mathcal{A}_d).$$

As the operators D and D^* given by,

$$D\eta = D^* \eta = \sqrt{b} \eta, \quad \eta \in L^2(\Gamma_1),$$

are bounded, Lemma 3.1 holds true. Thus in order to deduce the polynomial stability of the solution of (41), it is sufficient to check that the solution u_1 of (42) satisfies the observability inequality (O) ,

$$b \int_0^T \int_{\Gamma_1} |\eta_1^2| \gtrsim \|u_0\|_{\mathcal{D}(\mathcal{A}_c^{-2})}^2,$$

where $\mathcal{D}(\mathcal{A}_c^{-2})$ denotes throughout the example the space $(\mathcal{D}(\mathcal{A}_c^2))'$.

We first state the following proposition.

Lemma 6.7. *Let $u_0 = (y_0, \zeta_0, z_0, \xi_0, \eta_0)^\top \in \mathcal{H}$ and let $u_1 = (y_1, \zeta_1, z_1, \xi_1, \eta_1)^\top$ be the corresponding solution of the problem (42). Then there exists $c_T > 0$ such that*

$$(43) \quad \int_0^T \int_{\Gamma_1} |\eta_1^2| \geq c_T \|u_0\|_{\mathcal{D}(\mathcal{A}_c^{-2})}^2.$$

Proof. First step. Let $v_0 \in \mathcal{D}(\mathcal{A}_c)$ and $v = (y, \zeta, z, \xi, \eta)^\top$ be a solution of

$$(44) \quad v_t = \mathcal{A}_c v, \quad v(0) = v_0.$$

Then there exists $T > 0$ such that

$$(45) \quad \|v_0\|_{\mathcal{H}}^2 \leq C_1 \int_0^T \int_{\Gamma_1} |y_t|^2 + C_2 \int_0^T \int_{\Gamma_1} |\partial_\nu y|^2 + C_3 \int_0^T \int_{\Gamma_1} |\eta|^2,$$

for some positive constants C_1, C_2, C_3 .

Indeed, for $v_0 \in \mathcal{D}(\mathcal{A}_c)$, we have

$$(46) \quad \begin{aligned} \int_0^T \int_\Omega y_{tt}(2m \cdot \nabla y) &= - \int_0^T \int_\Omega y_t(2m \cdot \nabla y_t) + \left[\int_\Omega y_t 2m \cdot \nabla y \right]_0^T \\ &= 2 \int_0^T \int_\Omega |y_t|^2 - \int_0^T \int_\Gamma (m \cdot \nu) |y_t|^2 + \left[\int_\Omega y_t 2m \cdot \nabla y \right]_0^T, \end{aligned}$$

and

$$(47) \quad \begin{aligned} \int_0^T \int_\Omega \Delta y(2m \cdot \nabla y) &= - \int_0^T \int_\Omega \nabla y \cdot \nabla(2m \cdot \nabla y) + \int_0^T \int_\Gamma \partial_\nu y(2m \cdot \nabla y) \\ &= - \int_0^T \int_\Gamma (m \cdot \nu) |\nabla y|^2 + \int_0^T \int_\Gamma \partial_\nu y(2m \cdot \nabla y). \end{aligned}$$

Finally, multiplying the wave equation by $2m \cdot \nabla y$ and subtracting (47) from (46) leads to,

$$(48) \quad \begin{aligned} 2 \int_0^T \int_\Omega |y_t|^2 - \int_0^T \int_{\Gamma_1} (m \cdot \nu) |y_t|^2 + \int_0^T \int_\Gamma (m \cdot \nu) |\nabla y|^2 \\ + \left[\int_\Omega y_t 2m \cdot \nabla y \right]_0^T - \int_0^T \int_\Gamma \partial_\nu y(2m \cdot \nabla y) = 0. \end{aligned}$$

Multiplying the wave equation by y we obtain

$$(49) \quad - \int_0^T \int_\Omega |y_t|^2 + \int_0^T \int_\Omega |\nabla y|^2 + \left[\int_\Omega y_t y \right]_0^T - \int_0^T \int_\Gamma (\nu \cdot \nabla y) y = 0.$$

As

$$\int_0^T \int_\Gamma \partial_\nu y(2m \cdot \nabla y) = 2 \int_0^T \int_\Gamma (m \cdot \nu) (\partial_\nu y)^2 + 2 \int_0^T \int_\Gamma (m \cdot \tau) (\partial_\nu y \partial_\tau y),$$

then taking into consideration the Dirichlet condition on Γ_0 , we get

$$\begin{aligned} \int_0^T \int_\Gamma \partial_\nu y(2m \cdot \nabla y) - \int_0^T \int_\Gamma (m \cdot \nu) |\nabla y|^2 &= \int_0^T \int_\Gamma (m \cdot \nu) (\partial_\nu y)^2 - \int_0^T \int_{\Gamma_1} (m \cdot \nu) (\partial_\tau y)^2 \\ &\quad + 2 \int_0^T \int_{\Gamma_1} (m \cdot \tau) (\partial_\nu y \partial_\tau y). \end{aligned}$$

Due to the geometric conditions imposed on Γ , we have

$$(50) \quad \begin{aligned} \int_0^T \int_\Gamma \partial_\nu y(2m \cdot \nabla y) - \int_0^T \int_\Gamma (m \cdot \nu) |\nabla y|^2 &\leq \int_0^T \int_{\Gamma_1} (m \cdot \nu) (\partial_\nu y)^2 + \frac{R^2}{\omega} \int_0^T \int_{\Gamma_1} (\partial_\nu y)^2 \\ &\leq \left(R + \frac{R^2}{\omega}\right) \int_0^T \int_{\Gamma_1} (\partial_\nu y)^2. \end{aligned}$$

Hence (48) leads to

$$(51) \quad 2 \int_0^T \int_{\Omega} |y_t|^2 + \left[\int_{\Omega} y_t 2m \cdot \nabla y \right]_0^T \leq \int_0^T \int_{\Gamma_1} (m \cdot \nu) |y_t|^2 + (R + \frac{R^2}{\omega}) \int_0^T \int_{\Gamma_1} (\partial_{\nu} y)^2.$$

Adding (49) to (51), we obtain

$$(52) \quad \int_0^T \int_{\Omega} |y_t|^2 + \int_0^T \int_{\Omega} |\nabla y|^2 + \left[\int_{\Omega} y_t 2m \cdot \nabla y \right]_0^T + \left[\int_{\Omega} y_t y \right]_0^T - \int_0^T \int_{\Gamma} (\nu \cdot \nabla y) y \\ \leq \int_0^T \int_{\Gamma_1} (m \cdot \nu) |y_t|^2 + (R + \frac{R^2}{\omega}) \int_0^T \int_{\Gamma_1} (\partial_{\nu} y)^2.$$

Note moreover that $\left[\int_{\Omega} y_t 2m \cdot \nabla y \right]_0^T + \left[\int_{\Omega} y_t y \right]_0^T \gtrsim -E(0)$, and

$$\int_0^T \int_{\Gamma_1} \partial_{\nu} y y \leq \frac{1}{2\epsilon} \int_0^T \int_{\Gamma_1} (\partial_{\nu} y)^2 + \frac{\epsilon}{2} \int_0^T \int_{\Gamma_1} y^2 \leq \frac{1}{2\epsilon} \int_0^T \int_{\Gamma_1} (\partial_{\nu} y)^2 + \frac{c_p \epsilon}{2} \int_0^T \int_{\Omega} |\nabla y|^2.$$

We deduce that for $\epsilon > 0$ chosen small enough there exists $C > 0$ such that

$$(53) \quad (T - C) \|v_0\|_{\mathcal{H}}^2 - \frac{1}{a} \int_0^T \int_{\Gamma_1} |y_t|^2 - \int_0^T \int_{\Gamma_1} |\eta|^2 \leq \int_0^T \int_{\Gamma_1} (m \cdot \nu) |y_t|^2 + \frac{1}{2\epsilon} \int_0^T \int_{\Gamma_1} (\partial_{\nu} y)^2 \\ + (R + \frac{R^2}{\omega}) \int_0^T \int_{\Gamma_1} (\partial_{\nu} y)^2.$$

Finally, choosing T large enough, we get the required result (45).

Second step. Let $\alpha > 0$ and set

$$\eta_1 = \frac{1}{a} (-\partial_{\nu} y - \eta) + 2\alpha z + \alpha^2 \eta$$

We have the following expression for $|\eta_1|^2$ on Γ_1 ,

$$|\eta_1|^2 = \frac{1}{a^2} |\partial_{\nu} y|^2 + 4\alpha^2 |z|^2 + \frac{(a\alpha^2 - 1)^2}{a^2} |\eta|^2 - \frac{4\alpha}{a} \partial_{\nu} y z + \frac{2}{a^2} (1 - a\alpha^2) \partial_{\nu} y \eta + \frac{4\alpha}{a} (a\alpha^2 - 1) z \eta.$$

By the boundary condition on Γ_1 , $\eta_t = z$ and $\partial_{\nu} y = -\eta - a\eta_{tt}$, we get

$$|\eta_1|^2 = \frac{1}{a^2} |\partial_{\nu} y|^2 + 4\alpha^2 |z|^2 + \frac{(-1 + a\alpha^2)(1 + a\alpha^2)}{a^2} |\eta|^2 + 4\alpha^3 \eta \eta_t + 4\alpha \eta_t \eta_{tt} + \frac{2(-1 + a\alpha^2)}{a} \eta \eta_{tt}.$$

Thus

$$(54) \quad \int_0^T \int_{\Gamma_1} |\eta_1|^2 ds = \int_0^T \int_{\Gamma_1} \left[\frac{1}{a^2} |\partial_{\nu} y|^2 + (4\alpha^2 - \frac{2(-1 + a\alpha^2)}{a}) |z(1)|^2 + \frac{(-1 + a\alpha^2)(1 + a\alpha^2)}{a^2} |\eta|^2 \right] ds \\ + 2\alpha^3 \int_{\Gamma_1} (\eta(T)^2 - \eta(0)^2) ds + 2\alpha \int_{\Gamma_1} (\eta_t(T)^2 - \eta_t(0)^2) ds + \\ \frac{2(-1 + a\alpha^2)}{a} \int_{\Gamma_1} \eta(T) \eta_t(T) ds - \frac{2(-1 + a\alpha^2)}{a} \int_{\Gamma_1} \eta(0) \eta_t(0) ds.$$

Choosing α large enough, we get

$$w_1 = \frac{1}{a^2} > 0, w_2 = 4\alpha^2 - \frac{2(-1 + a\alpha^2)}{a} = \frac{2(1 + a\alpha^2)}{a} > 0, w_3 = \frac{(-1 + a\alpha^2)(1 + a\alpha^2)}{a^2} > 0.$$

In addition, (54) implies that

$$\int_0^T \int_{\Gamma_1} |\eta_1|^2 ds \geq \int_0^T \int_{\Gamma_1} [w_1 |\partial_\nu y|^2 + w_2 |z(1)|^2 + w_3 |\eta|^2] ds - K_{a,\alpha} \|v_0\|_{\mathcal{H}}^2,$$

for some constant $K_{a,\alpha} \geq 0$ independent of T .

Combining the previous inequality with (45), we deduce the existence of $c_1 > 0$ such that

$$\int_0^T \int_{\Gamma_1} |\eta_1|^2 ds \geq c_1(T-2) \|v_0\|_{\mathcal{H}}^2 - K_{a,\alpha} \|v_0\|_{\mathcal{H}}^2.$$

Finally, choosing T large enough, we obtain

$$(55) \quad \int_0^T \int_{\Gamma_1} |\eta_1|^2 ds = \int_0^T \int_{\Gamma_1} \left| \frac{1}{a} (-\partial_\nu y - \eta) + 2\alpha z(1) + \alpha^2 \eta \right|^2 ds \geq c_2 \|v_0\|_{\mathcal{H}}^2,$$

for some positive constant c_2 depending on T .

Last step. Let $u_0 \in D(\mathcal{A}_d)$ and let $u_1 = (y_1, \zeta_1, z_1, \xi_1, \eta_1)^\top$ be the corresponding solution of (42), then

$$v = (y, \zeta, z, \xi, \eta)^\top = [(\mathcal{A}_c + \alpha I)^2]^{-1} u,$$

is a solution of (44) where $v_0 = [(\mathcal{A}_c + \alpha I)^2]^{-1} u_0 \in \mathcal{D}(\mathcal{A}_c)$. Since $(\mathcal{A}_c + \alpha I)^2 = \mathcal{A}_c^2 + 2\alpha \mathcal{A}_c + \alpha^2 I$, the last component η_1 of u_1 is given by

$$\eta_1 = \frac{1}{a} (-\partial_\nu y - \eta) + 2\alpha z(1) + \alpha^2 \eta,$$

thus by the two previous steps we get (55). Noting that $\|u_0\|_{\mathcal{D}(\mathcal{A}_c)} \sim \|v_0\|_{\mathcal{H}}$, we consequently deduce that (43) holds for all $u_0 \in \mathcal{D}(\mathcal{A}_c)$. \square

Theorem 6.8. *Let $u_0 \in \mathcal{D}(\mathcal{A}_d)$ and let u be the solution of (41). Then u satisfies,*

$$(56) \quad \|u(t)\|^2 \lesssim \frac{1}{(1+t)^{\frac{1}{2}}} \|u_0\|_{\mathcal{D}(\mathcal{A}_d)}^2.$$

Proof. Since the operator $D \in \mathcal{L}(U)$, Lemma 3.1 holds true.

Set $\mathcal{H}_1 = D(\mathcal{A}_c)$ and $\mathcal{H}_2 = D(\mathcal{A}_c^{-2})$. Then $\mathcal{H} = [\mathcal{H}_1; \mathcal{H}_2]_{1/3}$. By Lemma 6.7, we have

$$\int_0^T \|D^* u_1(s)\|_{\mathcal{U}}^2 ds \geq c_T \|u_0\|_{\mathcal{H}_2}^2.$$

By Theorem 5.1 applied for $\theta = 1/3$, we therefore obtain (56). \square

Remark 6.9. *Using the same method we get an analogous result for the one dimensional problem. we can also get the observability inequality by a spectrum analysis and that was already done in the paper [5], where the authors obtained an optimal decay, thus we expect the decay in the two dimensional case to be optimal as well.*

Remark 6.10. *Consider the following system studied in [1]*

$$(57) \quad \begin{cases} y_{tt}(x, t) - y_{xx}(x, t) &= 0, & 0 < x < 1, t > 0, \\ y(0, t) &= 0, & t > 0, \\ y_x(1, t) + (\eta(t), C_0)_{\mathbb{C}^n} &= 0, & t > 0, \\ \eta_t(t) - B_0 \eta(t) - C_0 y_t(1, t) &= 0, & t > 0, \end{cases}$$

and

$$y(x, 0) = y_0(x), y_t(x, 0) = y_1(x), \eta(0) = \eta_0, 0 < x < 1,$$

where $B_0 \in M_n(\mathbb{C})$, $C_0 \in \mathbb{C}^n$ are given. System (57) can be written in the form (1) where $V = \{y \in H^1(0, 1) : y(0) = 0\}$, $X = L^2(0, 1)$ and $U = \mathbb{C}^n$. In this case, $\widehat{C} = B_0$ is a bounded operator and $B\eta = (\eta, C_0)\delta_1$ for all $\eta \in \mathbb{C}^n$. Indeed, since \widehat{C} is bounded then it is enough to verify assumption (O). Assumption (O) was verified in [1] and the polynomial stability of (57) was deduced. In particular, for $n = 1$ we obtain the system studied in [9], where a polynomial decay is proved using a multiplier method. The polynomial decay can be also obtained by proving an observability inequality for the solutions of the corresponding conservative system which is exactly what has been verified in [1], thus applying the approach introduced in this paper.

7 Unbounded example

Consider the following system

$$(58) \quad \begin{cases} u_{tt}(x, t) - u_{xx}(x, t) + w(x, t) = 0, & t \in [0, \infty), 0 < x < 1 \\ w_t(x, t) - iw_{xx}(x, t) + w(\xi, t)\delta_\xi - u_t(x, t) = 0, & t \in [0, \infty), 0 < x < 1 \\ u(0, t) = u(1, t) = w(0, t) = w(1, t) = 0, & t \in [0, \infty), \\ u(x, 0) = u_0(x), \partial_t u(x, 0) = u_1(x), w(x, 0) = w_0, & 0 < x < 1, \end{cases}$$

where $\xi \in (0, 1)$. Define the following spaces and operators:

$$X = U = L^2(0, 1), V = H_0^1(0, 1), U = L^2(0, 1), W = \mathbb{C},$$

$$A : u \in D(A) \rightarrow -u_{xx} \in L^2(0, 1), D(A) = H^2(0, 1) \cap H_0^1(0, 1),$$

and $B = B^* = I_U = I_{L^2(0, 1)}$. In addition,

$$D : \eta \in \mathbb{C} \rightarrow \eta\delta_\xi \in (D(C))', D^* : w \in D(C) \rightarrow w(\xi) \in \mathbb{C},$$

and

$$C : w \in D(C) \rightarrow -iw_{xx} \in L^2(0, 1), D(C) = H^2(0, 1) \cap H_0^1(0, 1).$$

The operator \widehat{C} is thus given by

$$\widehat{C}w = iw_{xx} - w(\xi)\delta_\xi$$

with

$$D(\widehat{C}) = \{w \in H_0^1(0, 1) \cap [H^2(0, \xi) \cap H^2(\xi, 1)] : i[w_x]_\xi = i[w_x(\xi^+) - w_x(\xi^-)] = w(\xi)\}.$$

As the operator D is unbounded, we need to verify that the problem satisfies assumption (H) as well as the assumption (O) for conservative problem. In this case we have

$$\mathcal{D}(\mathcal{A}_d) = D(A) \times V \times D(\widehat{C})$$

and

$$\mathcal{D}(\mathcal{A}_c) = D(A) \times V \times D(C).$$

In order to verify the assumption (H), we proceed by finding the transfer function, for this purpose we recall that $u_2 = u - u_1$ and $w_2 = w - w_1$ satisfies (15) which is in this case

$$(59) \quad \begin{cases} \partial_{tt}u_2 - \partial_{xx}u_2(x, t) + w_2(x, t) = 0, & t \in [0, \infty), 0 < x < 1 \\ \partial_t w_2(x, t) - i\partial_{xx}w_2(x, t) - \partial_t u_2 = w(\xi, t)\delta_\xi, & t \in [0, \infty), 0 < x < 1 \\ u_2(0, t) = u_2(1, t) = w_2(0, t) = w_2(1, t) = 0, \\ u_2(x, 0) = 0, \partial_t u_2(x, 0) = 0, w_2(x, 0) = 0, & 0 < x < 1, \end{cases}$$

and

$$(60) \quad \begin{cases} \partial_{tt}u_1 - \partial_{xx}u_1(x, t) + w_1(x, t) = 0, & t \in [0, \infty), 0 < x < 1 \\ \partial_t w_1(x, t) - i\partial_{xx}w_1(x, t) - \partial_t u_1 = 0, & t \in [0, \infty), 0 < x < 1 \\ u_1(0, t) = u_1(1, t) = w_1(0, t) = w_1(1, t) = 0, & t \in [0, \infty), \\ u_1(x, 0) = u_{1,0}, \partial_t u_1(x, 0) = u_{1,1}, w_1(x, 0) = w_{1,0}, & 0 < x < 1, \end{cases}$$

Verifying the assumption (H) is equivalent to verifying (see [4, Proposition 3.2] for more details)

$$|w_2(\xi, t)|^2 \lesssim |w(\xi, t)|^2.$$

For this purpose, we state the following proposition.

Proposition 7.1. *Let $(u_2, w_2) = (u - u_1, w - w_1)$ be the solution of (61). Then w_2 verifies*

$$|w_2(\xi, t)|^2 \leq |w(\xi, t)|^2.$$

Proof. Let $\lambda = 1 + i\eta$ and consider \hat{u}_2, \hat{w}_2 the Laplace transforms of u_2 and w_2 respectively. Then \hat{u}_2 and \hat{w}_2 satisfies (16) given by

$$(61) \quad \begin{cases} \lambda^2 \hat{u}_2(x, \lambda) - \partial_{xx} \hat{u}_2(x, \lambda) + \hat{w}_2(x, \lambda) = 0, \\ \lambda \hat{w}_2(x, \lambda) - i \partial_{xx} \hat{w}_2(x, \lambda) - \lambda \hat{u}_2 = \hat{w}(\xi, \lambda) \delta_\xi, \end{cases}$$

The problem reduces to studying \hat{u}_2 and \hat{w}_2 solutions of

$$(62) \quad \begin{cases} \lambda^2 \hat{u}_2 - \partial_{xx} \hat{u}_2 + \hat{w}_2 = 0 \\ \lambda \hat{w}_2 - i \partial_{xx} \hat{w}_2 - \lambda \hat{u}_2 = -i \delta_\xi \end{cases}$$

with

$$\hat{u}_2(0) = \hat{u}_2(1) = 0, \hat{w}_2(0) = \hat{w}_2(1) = 0, [\partial_x \hat{w}_2]_\xi = 1, [\hat{w}_2]_\xi = 0,$$

and proving the existence of $C_\beta > 0$

$$|w_2(\xi, \lambda)| \leq C_\beta, \forall \lambda = \beta + iy, y \in \mathbb{R}.$$

First, we set

$$\hat{w}_2 = \hat{w}_3 + \hat{w}_4,$$

where

$$(63) \quad \lambda \hat{w}_3 - i \partial_{xx} \hat{w}_3 = -i \delta_\xi,$$

with

$$(64) \quad \hat{w}_3(0) = \hat{w}_3(1) = 0, [\partial_x \hat{w}_3]_\xi = 1, [\hat{w}_3]_\xi = 0.$$

and

$$(65) \quad \lambda \hat{w}_4 - i \partial_{xx} \hat{w}_4 = \lambda \hat{u}_2.$$

with

$$(66) \quad \hat{w}_4(0) = \hat{w}_4(1) = 0.$$

Let $\beta > 0$ be fixed. It is required then to prove that

$$|\hat{w}_3(\xi, \lambda)| \leq C_{1\beta}, |\hat{w}_4(\xi, \lambda)| \leq C_{2\beta}, \forall \lambda = \beta + iy, y \in \mathbb{R}.$$

We start by writing the expression of \hat{w}_3 ,

$$\hat{w}_3(x, \lambda) = - \sum_{k=1}^{+\infty} \frac{\sqrt{2} \sin(k\pi\xi)}{k^2\pi^2 - i\lambda} \sqrt{2} \sin(k\pi x) = -2 \sum_{k=1}^{+\infty} \frac{|\sin(k\pi\xi)|^2}{k^2\pi^2 - i\lambda}.$$

For simplicity we consider $\lambda = 1 \pm i\pi^2 y^2$.

$$|\hat{w}_3(\xi, \lambda)| \lesssim \sum_{k=1}^{+\infty} \frac{1}{|(k^2 \pm y^2)\pi^2 - i|}.$$

We first give an estimate for $\lambda = 1 + i\pi^2 y^2$,

$$|\hat{w}_3(\xi, \lambda)| \lesssim \sum_{k=1}^{+\infty} \frac{1}{(k^2 + y^2)\pi^2} \leq \frac{1}{6}.$$

For $\lambda = 1 - i\pi^2 y^2$, we have

$$|\hat{w}_3(\xi, \lambda)| \leq \frac{2}{\pi^2} \left(\sum_{1 \leq k \leq E(y)-1} \frac{1}{y^2 - k^2} + 2\pi^2 + \sum_{E(y)+2 \leq k} \frac{1}{k^2 - y^2} \right).$$

But

$$\sum_{1 \leq k \leq E(y)-1} \frac{1}{y^2 - k^2} \leq \sum_{1 \leq k \leq E(y)-1} \frac{E(y) - k}{E(y)^2 - k^2} = \sum_{1 \leq k \leq E(y)-1} \frac{1}{E(y) + k} \leq 1.$$

and

$$\sum_{E(y)+2 \leq k} \frac{1}{k^2 - y^2} = \sum_{k=2}^{\infty} \frac{1}{(k + E(y))^2 - y^2} \leq \sum_{k=2}^{\infty} \frac{1}{(k-1)^2} = \frac{\pi^2}{6}.$$

Therefore $|\widehat{w}_3(\xi, \lambda)|$ is bounded on the line $\Re(\lambda) = 1$.

It remains to find the estimate satisfied by $\widehat{w}_4(\xi, \lambda)$. Indeed, since $(\sqrt{2} \sin(k\pi x))_{k \in \mathbb{N}^*}$ form a Hilbert basis of $L^2(0, 1)$, then we may write $\widehat{u}_2, \widehat{w}_2, \widehat{w}_4$ as follows

$$\widehat{u}_2(x, \lambda) = \sum_{k=1}^{+\infty} u_2^{(k)} \sqrt{2} \sin(k\pi x), \widehat{w}_2 = \sum_{k=1}^{+\infty} w_2^{(k)} \sqrt{2} \sin(k\pi x), \widehat{w}_4 = \sum_{k=1}^{+\infty} w_4^{(k)} \sqrt{2} \sin(k\pi x).$$

By the first equation of (62), we get

$$\forall k \geq 1, u_2^{(k)} = \frac{w_2^{(k)}}{k^2 \pi^2 + \lambda^2}.$$

Due to (65)

$$\forall k \geq 1, w_4^{(k)} = \frac{\lambda u_2^{(k)}}{ik^2 \pi^2 + \lambda}.$$

We deduce that

$$w_4^{(k)} = -\frac{\lambda w_2^{(k)}}{(k^2 \pi^2 + \lambda^2)(ik^2 \pi^2 + \lambda)}.$$

For $\lambda = 1 + iy$ we have

$$|k^2 \pi^2 + \lambda^2| = \sqrt{4y^2 + (1 + k^2 \pi^2 - y^2)^2} \geq 2|y|,$$

and

$$|ik^2 \pi^2 + \lambda| = |1 + ik^2 \pi^2 + iy| \geq |y|.$$

Hence for $|y|$ large enough we have

$$|w_4^{(k)}| \leq \frac{|w_2^{(k)}|}{|y|}.$$

Using $\widehat{w}_2 = \widehat{w}_3 + \widehat{w}_4$ we get for $|y|$ large enough

$$|w_4^{(k)}| \leq \frac{|w_3^{(k)}|}{|y|}.$$

We finally conclude that for $|y|$ large enough $|\widehat{w}_4(\xi, \lambda)|$ is bounded on the line $\Re(\lambda) = 1$. It follows that $|\widehat{w}_2(\xi, \lambda)|$ is bounded as well. □

In what follows we prove that the observability assumption (O) holds on subspaces of $\mathcal{D}(\mathcal{A}_d)$ on which we deduce the polynomial stability of the energy. Let us first remark that 0 is not an eigenvalue of \mathcal{A}_d . Let $\lambda = i\mu$ an eigenvalue of \mathcal{A}_d and $U = (u, v, w)$ a corresponding eigenvector. We then have,

$$(67) \quad \begin{cases} -\mu^2 u - \partial_{xx} u + w = 0 \\ \mu w - \partial_{xx} w - \mu u = iw(\xi) \delta_\xi. \end{cases}$$

with

$$u(0) = u(1) = w(0) = w(1) = 0.$$

Multiplying the second equation by \bar{w} then integrating by parts on $(0, 1)$, we find that $w(\xi) = 0$. We hence deduce that $w = 0$. Moreover, multiplying the first equation by \bar{u} , integrating by parts and considering the imaginary part we deduce that $u = 0$.

In order to verify the observability assumption (O) we study in what follows the spectrum of \mathcal{A}_c . Recall that the eigenvalues of \mathcal{A}_c are of the form $\lambda = i\mu, \mu \in \mathbb{R}$.

Proposition 7.2. *Let $\sigma(\mathcal{A}_c)$ be the set of eigenvalues of \mathcal{A}_c .*

(i) *Then every element of $\sigma(\mathcal{A}_c)$ is simple and $\sigma(\mathcal{A}_c)$ is a disjoint union of three sets:*

$$\sigma(\mathcal{A}_c) = \sigma_0 \cup \sigma_1 \cup \sigma_2$$

where σ_0 is a finite set, and there exists $k_0 \in \mathbb{N}^*$ such that $\sigma_1 = \{i\mu_{k,1}\}_{k \in \mathbb{Z}, |k| \geq k_0}$, and $\sigma_2 = \{i\mu_{k,2}\}_{k \in \mathbb{N}, k \geq k_0}$.

(ii) *For $i\mu_{k,i} \in \sigma_i, i = 1, 2$, an associated eigenvector $\phi_{\mu_{k,i}} = \frac{1}{|k|^{\alpha_i}}(u_{\mu_{k,i}}, v_{\mu_{k,i}}, w_{\mu_{k,i}})$, with $\alpha_1 = 1$ and $\alpha_2 = 4$ is given by*

$$u_{\mu_{k,i}}(x) = \sin(k\pi x), v_{\mu_{k,i}}(x) = i\mu_{k,i}u_{\mu_{k,i}}(x), w_{\mu_{k,i}}(x) = (\mu_{k,i}^2 - k^2\pi^2)\sin(k\pi x).$$

(iii) *The following estimates hold*

$$(68) \quad \mu_{k,1} = k\pi + \frac{1}{2\pi^2 k^2} + o\left(\frac{1}{k^2}\right), \quad |k| \rightarrow \infty,$$

$$(69) \quad \|\phi_{\mu_{k,1}}\|_{\mathcal{H}} \sim 1,$$

$$(70) \quad \mu_{k,1}^2 - k^2\pi^2 = \frac{1}{k\pi} + o\left(\frac{1}{k}\right),$$

$$(71) \quad \mu_{k,2} = -k^2\pi^2 + O\left(\frac{1}{k^2}\right), \quad k \rightarrow +\infty,$$

$$(72) \quad \|\phi_{\mu_{k,2}}\|_{\mathcal{H}} = O(1),$$

$$(73) \quad \mu_{k,2}^2 - k^2\pi^2 = k^4\pi^4 + O(k^2).$$

Proof. Let $\lambda = i\mu$ be an eigenvalue of \mathcal{A}_c and $U = (u, \lambda u, w)$ be a corresponding eigenvector of \mathcal{A}_c . Then u and w satisfies

$$(74) \quad \begin{cases} -\mu^2 u - \partial_{xx} u + w = 0 \\ \mu w - \partial_{xx} w - \mu u = 0. \end{cases}$$

Replacing w in the second equation, we find that

$$(75) \quad \begin{cases} \partial_{xxxx} u + (\mu^2 - \mu)\partial_{xx} u + (\mu - \mu^3)u = 0, \\ u(0) = \partial_{xx} u(0) = u(1) = \partial_{xx} u(1) = 0. \end{cases}$$

It is easy to check that $\mu = 0, \mu = 1$ and $\mu = -1$ are not eigenvalues of \mathcal{A}_c .

Let $X_1 = \frac{1}{2}(\mu - \mu^2 - \sqrt{\Delta})$ and $X_2 = \frac{1}{2}(\mu - \mu^2 + \sqrt{\Delta})$ be the roots of

$$p(X) = X^2 + (\mu^2 - \mu)X + \mu - \mu^3 = 0,$$

where $\Delta = \mu(\mu - 1)(\mu^2 + 3\mu + 4)$ is the discriminant of p .

Set $t_i = \sqrt{X_i}, i = 1, 2$ then the general form of u satisfying the first equation of (75) and the left boundary condition is

$$u(x) = c_1 \sinh(t_1 x) + c_2 \sinh(t_2 x).$$

Considering the right boundary conditions we see that u is non trivial if and only if t_1 and t_2 satisfy

$$\sinh(t_1) \sinh(t_2)(t_1^2 - t_2^2) = 0.$$

But $t_1^2 - t_2^2 \neq 0$, since $\mu \neq 0$ and $\mu \neq 1$. Hence t_1 and t_2 satisfy the following characteristic equation

$$\sinh(t_1) \sinh(t_2) = 0,$$

which gives that $t_1 = ik\pi$ or $t_2 = ik\pi, k \in \mathbb{Z}^*$ i.e $X_1 = -k^2\pi^2$ or $X_2 = -k^2\pi^2$.

Now, we remark that all the eigenvalues of \mathcal{A}_c are simple. Suppose otherwise that there exists a double eigenvalue, then there exist $k_i, i = 1, 2$ s.t $X_i = -k_i\pi^2, i = 1, 2$. Thus we have

$$\frac{X_1 X_2}{X_1 + X_2} = -\frac{k_1^2 k_2^2 \pi^2}{k_1^2 + k_2^2} = \mu + 1.$$

Now, replacing μ in $X_1 + X_2 = \mu - \mu^2$, we find that

$$2k_1^4 + 4k_1^2 k_2^2 + 2k_2^4 - k_1^6 \pi^2 - k_2^6 \pi^2 + k_1^4 k_2^4 \pi^4 = 0,$$

which is impossible since π^2 is a transcendental number.

Therefore,

$$u(x) = \sin(k\pi x), \quad w(x) = (\mu^2 + X_i) \sin(k\pi x), \quad i = 1 \text{ or } 2.$$

Moreover, the eigenvalues of \mathcal{A}_c are formed of two disjoint families of eigenvalues. The first class of eigenvalues is obtained from $X_1 = -k^2\pi^2$, the second class is obtained from $X_2 = -k^2\pi^2$.

Now, we firstly study the asymptotic behaviour of the first class: since $X_1 = -\mu^2 + \frac{1}{\mu} + o(\frac{1}{\mu}) = -k^2\pi^2$ then $\mu = k\pi + \frac{1}{2\pi^2 k^2} + o(\frac{1}{k^2}), |k| \rightarrow \infty$. If we denote by $\{i\mu_{k,1}\}_{k \in \mathbb{Z}^*}$ this first class of eigenvalues then the previous estimate is (68). Using the previous estimate we directly get (69) and (70).

Secondly, since $X_2 = \mu + O(\frac{1}{\mu}) = -k^2\pi^2$ we deduce that the large eigenvalues of the second class are negative, and denoting them by $i\mu_{k,2}$ we easily see that (71) holds true. Moreover, since $\mu_{k,2}^2 - k^2\pi^2 = O(k^4)$ then (72) holds. \square

In order to use generalized Inghams inequalities we need to estimate $\inf_{\mu_{k,1} \in \sigma_1, \mu_{k',2} \in \sigma_2} |\mu_{k,1} - \mu_{k',2}|$. Unfortunately it seems to be a difficult task and it remains a open question. Hence, to get an observability result we will take the initial condition U_0 in some subspaces of \mathcal{H} . So we introduce

$$H_1 = \text{span}(\phi_\mu)_{\mu \in \sigma_0} \cup \text{span}(\phi_\mu)_{\mu \in \sigma_1} \text{ and } H_2 = \text{span}(\phi_\mu)_{\mu \in \sigma_0} \cup \text{span}(\phi_\mu)_{\mu \in \sigma_2}.$$

Before given an observability result we introduce the set \mathcal{S} of all numbers $\rho \in (0, \pi)$ such that $\frac{\rho}{\pi} \notin \mathbb{Q}$ and if $[0, a_1, \dots, a_n, \dots]$ is the expansion of $\frac{\rho}{\pi}$ as a continued fraction, then (a_n) is bounded. Recall that if $\pi\xi \in \mathcal{S}$ then

$$(76) \quad |\sin(k\pi\xi)| \gtrsim \frac{1}{|k|}, \quad k \in \mathbb{Z}^*,$$

(see for instance [3]).

Proposition 7.3. 1. For all $\xi \in (0, 1)$ there are not $T, C > 0$ such that for all $U_0 \in \mathcal{H}$ we have

$$(77) \quad \int_0^T |w(\xi, t)|^2 dt \geq C_T \|U_0\|_{\mathcal{H}}^2.$$

2. Suppose that $\xi \in \mathcal{S}$.

Let $U_0 \in H_1$ and $U = (u, v, w)$ be the corresponding solution of the conservative problem

$$(78) \quad U_t = \mathcal{A}_c U, U(0) = U_0.$$

Then there exists $T > 0$ and a constant $c_T > 0$ such that

$$(79) \quad \int_0^T |w(\xi, t)|^2 dt \geq C_T \|U_0\|_{\mathcal{D}(\mathcal{A}_c^{-3})}^2,$$

where $\mathcal{D}(\mathcal{A}_c^{-3}) = (\mathcal{D}(\mathcal{A}_c^3))'$, obtained by means of the inner product in X .

For $U_0 \in H_2$ we have

$$(80) \quad \int_0^T |w(\xi, t)|^2 dt \geq C_T \|U_0\|_{\mathcal{D}(\mathcal{A}_c^{-\frac{1}{2}})}^2.$$

Proof. 1. Since

$$\lim_{n \rightarrow +\infty} \|(i\mu_{n,1} - \mathcal{A}_c) \phi_{n,1}\|_{\mathcal{H}}^2 + \left\| \begin{pmatrix} 0 & 0 & D^* \end{pmatrix} \phi_{n,1} \right\|_U^2 = 0.$$

Which implies according to [6, Theorem 5.1] that we don't have the exact observability, i.e., the inequality (77).

2. Let $U_0 \in H_1$. We may write

$$U_0 = \sum_{\mu \in \sigma_0} u_0^\mu \phi_\mu + \sum_{|k| \geq k_0} u_0^{(k)} \phi_{\mu_{k,1}}.$$

Moreover,

$$w(\xi, t) = \frac{1}{|k|} \left(\sum_{\mu \in \sigma_0} u_0^\mu e^{i\mu t} w_\mu(\xi) + \sum_{|k| \geq k_0} u_0^{(k)} e^{i\mu_{k,1} t} w_{\mu_{k,1}}(\xi) \right).$$

Note that $\gamma_1 = \inf_{\mu, \mu' \in \sigma, \mu \neq \mu'} |\mu - \mu'| > 0$, then using Ingham's inequality there exists $T > 2\pi\gamma_1 > 0$ and a constant $c_T > 0$ depending on T such that

$$\int_0^T |w_1(\xi, t)|^2 dt \geq c_T \frac{1}{|k|} \left(\sum_{\mu \in \sigma_0} |u_0^\mu w_\mu(\xi)|^2 + \sum_{|k| \geq k_0} |u_0^{(k)} w_{\mu_{k,1}}(\xi)|^2 \right).$$

Now using (ii), and estimates (68), (69), (70) of Proposition 7.2 we get (79). For $U_0 \in H_2$, we use analogous argument. □

Theorem 7.4. 1. For any $\xi \in (0, 1)$, the system described by (58) is not exponentially stable in \mathcal{H} .

2. Let $U_0 \in H_1 \cap \mathcal{D}(\mathcal{A}_d)$, and let U be the solution of the corresponding dissipative problem

$$U_t = \mathcal{A}_d U, \quad U(0) = U_0.$$

Then U satisfies,

$$(81) \quad \|U(t)\|^2 \lesssim \frac{1}{(1+t)^{\frac{1}{3}}} \|U_0\|_{\mathcal{D}(\mathcal{A}_d)}^2.$$

3. Let $U_0 \in H_2 \cap \mathcal{D}(\mathcal{A}_d)$, and let U be the solution of the corresponding dissipative problem

$$U_t = \mathcal{A}_d U, \quad U(0) = U_0.$$

Then U satisfies,

$$(82) \quad \|U(t)\|^2 \lesssim \frac{1}{(1+t)^2} \|U_0\|_{\mathcal{D}(\mathcal{A}_d)}^2.$$

Proof. 1. This result is a direct consequence of the first assertion of Proposition 7.3 and Theorem 4.1.

2. Due to Proposition 7.1 and Proposition 7.3 we deduce (81) from Theorem 5.1 setting $\mathcal{H}_1 = \mathcal{D}(\mathcal{A}_c)$ and $\mathcal{H}_2 = \mathcal{D}(\mathcal{A}_c^{-3})$ and $\theta = \frac{1}{4}$.

3. As in 2. we deduce (82) setting $\mathcal{H}_1 = \mathcal{D}(\mathcal{A}_c)$ and $\mathcal{H}_2 = \mathcal{D}(\mathcal{A}_c^{-\frac{1}{2}})$ and $\theta = \frac{2}{3}$. □

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